Trends in Mathematics
Information Center for Mathematical Sciences
Volume 2, Number 1, September 1999, Pages 1–

DIRECT SUMS OF IRREDUCIBLE OPERATORS

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ABSTRACT. It is known that every operator on a (separable) Hilbert space is the direct integral of irreducible operators, but not every one is the direct sum of irreducible ones. We show that an operator can have either finitely or uncountably many reducing subspaces, and the former holds if and only if the operator is the direct sum of finitely many irreducible operators no two of which are unitarily equivalent. We also characterize operators T which are direct sums of irreducible operators in terms of the C^*-structure of the commutant of the von Neumann algebra generated by T.

1. INTRODUCTION

A bounded linear operator on a complex separable Hilbert space H is *irreducible* if it has no reducing subspace other than $\{0\}$ and H; otherwise, it is *reducible*. In this paper, we are concerned with the problem of characterizing operators which are expressible as the direct sum of irreducible operators. Examples of such operators include any finite-dimensional operator, compact operator, completely nonnormal essentially normal operator, completely nonnormal hyponormal operator with finite multiplicity (cf. [7, Section 2.1]) and any Cowen-Douglas operator (cf. [3, Prop. 1.18]). On the other hand, not every operator can be expressed as such a direct sum. This is the case even for normal operators since it can be easily seen that a normal operator is irreducible if and only if it acts on a one-dimensional space, and thus it is the direct sum of irreducible operators if and only if it is diagonalizable. In particular, the bilateral shift (the operator of multiplication by the independent variable on the L^2-space of the unit circle) cannot be the direct sum of irreducible operators.

In Section 2 below, we first show in Theorem 2.1 that no operator can have countably infinitely many reducing subspaces, that is, the number of reducing subspaces of any operator is either finite or $\aleph_1$, the cardinal number of real numbers. Moreover, an operator has finitely many reducing subspaces if and only if it is the direct sum of finitely many irreducible operators no two of which are unitarily equivalent. These are proved by making use of the structure theorem of two projections (Lemma 2.2).

An equivalent condition for irreducibility can be formulated in terms of the von Neumann algebra generated by the operator. Indeed, if $W^*(T)$ denotes the von

1991 *Mathematics Subject Classification.* 47A15, 47C15.
Key words and phrases. Irreducible operator, reducing subspace, von Neumann algebra.

Neumann algebra generated by an operator T on H and $W^*(T)'$ denotes its commutant, then using the von Neumann double commutant theorem we can easily show the equivalence of the following three conditions: (1) T is irreducible, (2) dim $W^*(T)'=1$, and (3) $W^*(T)$ equals $\mathcal{B}(H)$, the algebra of all operators on H. In Section 3, we will generalize this to the situation for direct sums of irreducible operators. We show in Theorem 3.1 that T is such a direct sum if and only if $W^*(T)'$ is $*$-isomorphic to the direct sum of full matrix algebras $M_{n_i}(\mathbf{C})$ with various sizes n_i, $1 \leq n_i \leq \infty$. Here $M_{n_i}(\mathbf{C})$, $1 \leq n_i \leq \infty$, denotes the algebra of all n_i-by-n_i complex matrices, and $M_\infty(\mathbf{C})$ is understood to be $\mathcal{B}(l^2)$. As a corollary (Corollary 3.2), we have the equivalence of T being the direct sum of finitely many irreducible operators and dim $W^*(T)' < \infty$.

We conclude this section with two further remarks. Firstly, it is known that on an infinite-dimensional separable Hilbert space H, there are plenty of irreducible operators in the sense that such operators are dense in $\mathcal{B}(H)$ in the norm topology (cf. [4]). In [4], it was asked whether reducible operators are also dense. This is answered positively by Voiculescu [10]. In fact, an even stronger result is true, namely, for any operator T and any $\varepsilon > 0$, there is a compact operator K with $||K|| < \varepsilon$ such that $T+K$ is the direct sum of infinitely many irreducible operators (cf. also [6, Prop. 4.21 (iv) and (v)]).

Secondly, although not every operator is the direct sum of irreducible operators, every one can be decomposed as the direct integral of irreducible ones. This is what the next proposition says.

Proposition 1.1. *Every operator is the direct integral of irreducible operators.*

This is an easy consequence of [1, Theorem 3.6] on the direct integral decomposition of operator algebras.

2. NUMBER OF REDUCING SUBSPACES

The main result of this section is the following theorem.

Theorem 2.1. *The number of reducing subspaces of any operator is either finite or uncountably infinite. It is the former case if and only if the operator is the direct sum of finitely many irreducible operators $\sum_{i=1}^n \oplus T_i$ with T_i and T_j non-unitarily-equivalent for any $i \neq j$. In this case, the number of reducing subspaces is 2^n.*

This has an analogue in a different context: the number of invariant subspaces of any operator on a finite-dimensional space is either finite or uncountably infinite, and it is the former case if and only if the operator is cyclic (cf. [8]).

The proof of Theorem 2.1 is based on three lemmas. The first one is a structure theorem for two arbitrary (orthogonal) projections. This result has appeared repeatedly in the literature before; the version we adopt below is from [5].

Lemma 2.2. *Let P and Q be arbitrary two projections on a Hilbert space. Then there is a unitary operator U such that*

$$U^*PU = \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix} \oplus I_2 \oplus I_3 \oplus 0 \oplus 0$$

and

$$U^*QU = \begin{pmatrix} A & B \\ B & I_1 - A \end{pmatrix} \oplus I_2 \oplus 0 \oplus I_4 \oplus 0$$

on the space $H_1 \oplus H_1 \oplus H_2 \oplus H_3 \oplus H_4 \oplus H_5$, where A is a positive contraction on H_1 and B is the positive square root of $A(I_1 - A)$. We may require that $0 < A \leq \frac{1}{2} I_1$, in which case A is unique up to unitary equivalence.

The preceding lemma is used to prove

Lemma 2.3. *If T has countably many reducing subspaces, then $W^*(T)'$ is abelian.*

The next lemma is elementary.

Lemma 2.4. *Let A and B be irreducible operators on H and K, respectively. Then A and B are unitarily equivalent if and only if there is a nonzero operator X such that $XA = BX$ and $XA^* = B^*X$.*

3. FULL MATRIX ALGEBRAS

In this section, we will characterize the direct sum of irreducible operators in terms of the C^*-algebra structure of the commutant of its generated von Neumann algebra.

For any operator T on H and any integer n, $1 \leq n \leq \infty$, let $T^{(n)}$ denote the direct sum of n copies of T on $H^{(n)} = \underbrace{H \oplus \ldots \oplus H}_{\text{n}}$.

Theorem 3.1. *An operator T on H is the direct sum of irreducible operators, say, $\sum_{i=1}^{n} \oplus T_i^{(n_i)}$ on $\sum_{i=1}^{n} \oplus H_i^{(n_i)}$, where $1 \leq n \leq \infty, 1 \leq n_i \leq \infty$ for all i and the T_i's are pairwise non-unitarily-equivalent, if and only if $W^*(T)'$ is $*$-isomorphic to $\sum_{i=1}^{n} \oplus M_{n_i}(\mathbf{C})$. Moreover, the T_i's are unique up to permutation and unitary equivalence. More precisely, if $T = \sum_{k=1}^{m} \oplus S_k^{(m_k)}$ is another direct sum representation of irreducible operators for T with pairwise-non-unitarily-equivalent S_k's, then $n = m$ and there is a permutation π of $\{1, \ldots, n\}$ and a unitary operator U in $W^*(T)'$ such that $n_i = m_{\pi(i)}$ and $UT_i = S_{\pi(i)}U$ for all i.*

Since every finite-dimensional (unital) C^*-algebra is $*$-isomorphic to the direct sum of finitely many full (finite) matrix algebras (cf. [9, Theorem 11.2]), an easy consequence of the preceding theorem is

Corollary 3.2. *T is the direct sum of finitely many irreducible operators if and only if* $\dim W^*(T)' < \infty$

We need the following lemmas for the proof of Theorem 3.1.

Lemma 3.3. *If T is irreducible on H and X is such that $XT = TX$ and $XT^* = T^*X$, then X is a scalar operator.*

Lemma 3.4. *Let P be a projection in $W^*(T)'$. Then $T \mid (\mathrm{ran}\ P)$ is irreducible if and only if P is a minimal projection in $W^*(T)'$.*

Recall that a projection p in a C^*-algebra is *minimal* if there is no projection q, other than 0 and p, such that $pq = q$.

Lemma 3.4 is an easy consequence of the definitions of irreducibility and minimal projection.

We next consider the problem when two operators have isomorphic reducing subspace lattices. When the operators are normal, this has been solved by Conway and Gillespie [2]. Using their result, we may settle the problem when the two operators are both direct sums of irreducible ones. This covers in particular the cases for operators on finite-dimensional spaces and compact operators.

For any operator T, let Red T denote the lattice of its reducing subspaces.

Proposition 3.5. *Let $A = \sum_{j=1}^{n} \oplus {A_j}^{(n_j)}$ and $B = \sum_{k=1}^{m} \oplus {B_k}^{(m_k)}$ be direct sums of irreducible operators with pairwise non-unitarily-equivalent A_j's and B_k's, where $1 \le n, m \le \infty$ and $1 \le n_j, m_k \le \infty$ for all j and k, and the n_j's and m_k's are decreasing. Then* Red A *is isomorphic to* Red B *if and only if $n = m$ and $n_j = m_j$ for all j.*

4. Acknowledgements

The research of the first two authors was supported by National Natural Science Foundation of China and Mathematical Center of State Education Commission of China while that of the third author by the Natioinal Science Council of the Republic of China (Taiwan).

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