Trends in Mathematics
Information Center for Mathematical Sciences
Volume 2, Number 1, September 1999, Pages 1–

EQUILIBRIUM THEOREMS OF MULTIMAPS AND FUZZY MAPPINGS

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ABSTRACT. In this paper, we first establish some existence theorems of equilibrium problem for multimaps, then use these theorems to establish some existence theorems of equilibrium problem for fuzzy maps.

1. INTRODUCTION

Let X be a convex subset of a real topological vector space E, Z be a real topological vector space and let $F : X \times X \to \mathcal{F}(Z)$ and $c : X \to \mathcal{F}(Z)$ be two fuzzy mappings(where $\mathcal{F}(Z)$ denotes the collection of all fuzzy set on Z), then $F(x,y)$(denoted by $F_{x,y}$) and $c(x)$(denoted by c_x) are fuzzy set in $\mathcal{F}(Z)$ for all $x, y \in X$. Let $\alpha : X \to (0,1]$ be a map, then $(F_{x,y})_{\alpha(x)}$ and $(c_x)_{\alpha(x)}$ denote the $\alpha(x)$-cut set of $F_{x,y}$ and c_x respectively, the equilibria of fuzzy mappings is the problem of finding $\overline{x} \in X$ such that

problem (1) $(F_{\overline{x},y})_{\alpha(\overline{x})} \not\subset (c_{\overline{x}})_{\alpha(\overline{x})}$ for all $y \in X$.

or problem (2) $(F_{\overline{x},y})_{\alpha(\overline{x})} \cap (c_{\overline{x}})_{\alpha(\overline{x})} = \emptyset$ for all $y \in X$.

This type of problems contain many problem as special cases.

Example 1. Let $S : X \times X \multimap Z$ and $T : X \multimap Z$ be two given multimaps and $\alpha : X \to (0,1]$ be a function. Now we define two fuzzy mappings F and c by

$$F : X \times X \to \mathcal{F}(Z) \qquad (x,y) \to \chi_{S(x,y)}$$
$$c : X \to \mathcal{F}(Z) \qquad x \to \chi_{T(x)},$$

Where $\chi_U(\cdot)$ is a characteristic function of of a set U. If $\overline{x} \in X$ is a solution of the problem

$$(F_{\overline{x},y})_{\alpha(\overline{x})} \not\subset (c_{\overline{x}})_{\alpha(\overline{x})} \qquad \text{for all } y \in X.$$
$$(\text{ resp. } \quad (F_{\overline{x},y})_{\alpha(\overline{x})} \cap (c_{\overline{x}})_{\alpha(\overline{x})} = \emptyset \text{ for all } y \in X. \quad)$$

Then

$$S(\overline{x},y) \not\subset T(\overline{x}) \qquad \text{for all } y \in X. \tag{3}$$
$$(\text{ resp. } \quad S(\overline{x},y) \cap T(\overline{x}) = \emptyset \text{ for all } y \in X. \quad) \tag{4}$$

Problems (3) and (4) are problems of equilibrium of multimap recently study by Oettli and Schläger [8, 9, 10]. Hence problems (1) and (2) contains problems (3) and (4) as special case respectively. If $f : X \times X \to \mathbb{R}$ is a function and if $S(x,y) = \{f(x,y)\}$ and $T(x) = (-\infty, 0)$. Then (3) and (4) reduces to the problem of finding $\overline{x} \in X$ such that

problem (5) $f(\overline{x},y) \geq 0$ for all $y \in X$.

Problem (5) is a scalar equilibrium in the sense of Blum and Oettli [1].This type of problem contains optimization problem, problem of the Nash type equilibria, complementary problems, fixed point problems, variational inequalities problems and many others as special cases.

Recently, Chang et al [2, 3] established some properties of fuzzy mappings and existence theorems of generalized fuzzy vector variational inequalities. Chang [4] established coincidence theorem for fuzzy mappings. Chang [5] established fixed point theorems for fuzzy mappings.

In this paper, we first apply a fixed point theorem of Lin and Yu [6] to establish the existence theorems of problem (3) and (4). Then we apply these results and properties of fuzzy mappings established by Chang et al [2, 3] to establish the existence theorems of problem (1) and (2).

2. Preliminaries

Let X, Z be two topological vector spaces. A fuzzy set on Z is a function with domain Z and values in $[0,1]$, in the sequel we shall always denote the collection of all fuzzy set on Z by $\mathcal{F}(Z)$. A mapping F from X into $\mathcal{F}(Z)$ is called fuzzy mapping. If $F : X \to \mathcal{F}(Z)$ is a fuzzy mapping, then $F(x)$(denoted by F_x) is a fuzzy set on Z and $F_x(z)$, $z \in Z$ is called the degree of membership of z in F_x.

Definition 1.[3]. Let $F : X \to \mathcal{F}(Z)$ be a fuzzy mapping. F is said to be convex if for any $x \in X$, $y, z \in Z$ and $t \in [0,1]$,

$$F_x(ty + (1-t)z) \geq min\{F_x(y), F_x(z)\}.$$

Definition 2.[3]. Let $F : X \to \mathcal{F}(Z)$ be a fuzzy map. F is said to be closed if $F_x(z)$ is upper semicontinuous on $X \times Z$ as a real ordinary function.

Definition 3.[3]. Let $F : X \to \mathcal{F}(Z)$ be a fuzzy map. We say that
(i)F is topologically open at $x_0 \in X$ if for each open subset U of Z with $F_{x_0}(z) \geq \alpha$ for some $z \in U (\alpha \in (0,1])$, there is a neighborhood V of x_0 in X such that if $x \in V$, then $F_x(z) \geq \alpha$ for some $z \in U$.
(ii)F is topologically closed at $x_0 \in X$ if for each open subset U of Z such that $F_{x_0}(z) \geq \alpha$, then $z \in U (\alpha \in (0,1])$, there is a neighborhood V of x_0 in X such that if $x \in V$ and $F_x(z) \geq \alpha$, then $z \in U$.
Here we say F is topologically open(closed) if F is topologically open(closed) at x for all $x \in X$.

Definition 4. Let A be a fuzzy set in $\mathcal{F}(Z)$ and $\alpha \in (0,1]$. Then the set
(i)$(A)_\alpha = \{z \in Z : A(z) \geq \alpha\}$ is called an α-cut set of A.
(ii)$(A)^\alpha = \{z \in Z : A(z) > \alpha\}$ is called a strong α-cut set of A.

Definition 5.[12]. Let A be a fuzzy set in $\mathcal{F}(Z)$. A is said to be compact if for each $\alpha \in (0,1]$, the α-cut set $(A)_\alpha$ is compact in Z.

Lemma A.[2]. Let $F : X \to \mathcal{F}(Z)$ be a convex fuzzy mapping,$\alpha : X \to (0,1]$ be a function and $f : X \multimap Z$ be defined by $f(x) = (F_x)_{\alpha(x)}$.
Then $f(x)$ is a convex set in Z for all $x \in X$.

Lemma B.[2]. Let $F : X \to \mathcal{F}(Z)$ be a closed fuzzy mapping,$\alpha : X \to (0,1]$

be a function and $f : X \multimap Z$ be defined by $f(x) = (F_x)_{\alpha(x)}$.
Then $f(x)$ is a closed set in Z for all $x \in X$.

Definition 6. Let X and Y be two topological spaces, $F : X \multimap Y$. We say that
(i)F is upper semicontinious at $x_0 \in X$ if for each open set U of Y with $F(x_0) \subset U$, there is an neighborhood V of x_0 in X such that if $x \in V$, then $F(x) \subset U$.
(ii)F is lower semicontinious at $x_0 \in X$ if for each open set U of Y with $F(x_0) \cap U \neq \emptyset$, there is an neighborhood V of x_0 in X such that if $x \in V$, then $F(x) \cap U \neq \emptyset$.
Here we say F is upper(lower) semicontinious if F is upper(lower) semicontinious at x for all $x \in X$.

Definition 7.[11]. Let X and Y be two topological spaces, $F : X \multimap Y$. F is said to be transfer open if for any $x \in X$ and any $y \in F(x)$, there exists an $\overline{x} \in X$ such that $y \in intF(\overline{x})$.

The following Lemma follows immediately from Definition 7.
Lemma C. Let X and Y be two topological spaces, $F : X \multimap Y$, then the following stasements are equivalent.
(i)$F^- : Y \multimap X$ is transfer open and for any $x \in X$, $F(x)$ is nonempty.
(ii)$X = \bigcup\{intF^-(y) : y \in Y\}$.

3. Main results

Theorem A.[6]. Let X be a convex space and $F : X \multimap X$ a nonempty map such that
(i) $X = \bigcup\{int(F^-(y)) : y \in X\}$; and
(ii) there exists a nonempty compact subset K of X such that for each $M \in \langle X\rangle$, there exists a compact convex subset L_M of X containing M such that

$$L_M \cap \bigcap\{X \setminus int(coF)^-(y) : y \in L_M\} \subset K.$$

Then there exists a $\overline{x} \in X$ such that $\overline{x} \in coF(\overline{x})$.

Now we establish some existence theorems for equilibrium problem by using theorem A.

Theorem 1. Let X be a convex subset of a topological vector space E, Z be a real topological vector space and C, $D : X \multimap Z$; f, $g : X \times X \multimap Z$ satisfy the following conditions:
(i)for all $x, y \in X$, $f(x, y) \not\subset C(x)$ implies $g(x, y) \subset D(y)$;
(ii)$G^- : X \multimap X$ is transfer open, where $G : X \multimap X$ is defined by $G(x) = \{y \in X : g(x, y) \not\subset D(y)\}$, $x \in X$;
(iii) for all $x \in X$, $\{y \in X : f(x, y) \subset C(x)\}$is convex;
(iv) for all $x \in X$, if $g(x, y) \subset D(y)$ for all $y \in X$, then $f(x, y) \not\subset C(x)$ for all $y \in X$;
(v) for all $x \in X$, $f(x, x) \not\subset C(x)$; and
(vi)there exists a nonempty compact subset K of X such that for all $M \in \langle X\rangle$, there exists a compact convex subset L_M of X containing M such that

$$L_M \cap \bigcap\{X \setminus intF^-(y) : y \in L_M\} \subset K,$$

where $F : X \multimap X$ is defined by $F(x) = \{y \in X : f(x,y) \subset C(x)\}$.
Then there exists an $\overline{x} \in X$ such that $f(\overline{x}, y) \not\subset C(\overline{x})$ for all $y \in X$.

Remark :(1) Theorem 1 generalizes Theorem 1[8]. In Theorem 1, X is not assume to be compact, G^- is transfer open which is more general than open values. Condition (vi) of Theorem 1 is weaker than condition (vi) of Theorem 1[8]. In fact, in condition (vi) of Theorem 1[8], if we let $K = A$ and for all $M \in \langle X \rangle$ let $L_M = co\{B \cup M\}$, then condition (vi) of Theorem 1 hold clearly.
(2) In Theorem 3[7], we assume that for each $x \in X$, $G(x) = \{y \in X : g(x,y) \not\subset D(y)\}$ is $G-$convex, while in Theorem 1, we assume that for each $x \in X$, the set $\{y \in X : f(x,y) \subset C(x)\}$ is convex and the proof has some difference.
Applying Theorem A and following the same argument as Theorem 1, we have

Theorem 2. Let X be a convex subset of a topological vector space E, Z be a real topological vector space and $C, D : X \multimap Z$; $f, g : X \times X \multimap Z$ satisfy the following conditions:
(i)for all $x, y \in X$, $f(x,y) \cap C(x) = \emptyset$ implies $g(x,y) \cap D(y) \neq \emptyset$;
(ii)$G^- : X \multimap X$ is transfer open, where $G : X \multimap X$ is defined by $G(x) = \{y \in X : g(x,y) \cap D(y) = \emptyset\}$, $x \in X$;
(iii) for all $x \in X$, $\{y \in X : f(x,y) \cap C(x) \neq \emptyset\}$is convex;
(iv) for all $x \in X$, if $g(x,y) \cap D(y) \neq \emptyset$ for all $y \in X$, then $f(x,y) \cap C(x) = \emptyset$ for all $y \in X$;
(v) for all $x \in X$, $f(x,x) \cap C(x) = \emptyset$; and
(vi)there exists a nonempty compact subset K of X such that for all $M \in \langle X \rangle$, there exists a compact convex subset L_M of X containing M such that

$$L_M \cap \bigcap \{X \setminus intF^-(y) : y \in L_M\} \subset K,$$

where $F : X \multimap X$ is defined by $F(x) = \{y \in X : f(x,y) \cap C(x) \neq \emptyset\}$.
Then there exists an $\overline{x} \in X$ such that $f(\overline{x}, y) \cap C(\overline{x}) = \emptyset$ for all $y \in X$.

The following Lemma is need in this paper.

Lemma 1. Let X be a convex set, $f : X \times X \multimap Z$, and $C : X \multimap Z$ be a multimap such that for all $x \in X$, $C(x)$ is convex.
(i)if for all $x \in X$, $f(x, \beta v + (1-\beta)w) \subset \beta f(x,v) + (1-\beta)f(x,w)$ for all $v, w \in X$ and $\beta \in [0,1]$, then for all $x \in X$, $\{y \in X : f(x,y) \subset C(x)\}$ is convex;
(ii)if for all $x \in X$, $\beta f(x,v) + (1-\beta)f(x,w) \subset f(x, \beta v + (1-\beta)w)$ for all $v, w \in X$ and $\beta \in [0,1]$, then for all $x \in X$, $\{y \in X : f(x,y) \cap C(x) \neq \emptyset\}$ is convex.

From Lemma 1 and Theorem 1, we have the following Theorem.

Theorem 3. Let X be a convex subset of a topological vector space E, Z be a real topological vector space and $C, D : X \multimap Z$; $f, g : X \times X \multimap Z$ satisfy the following conditions:
(i)for all $x, y \in X$, $f(x,y) \not\subset C(x)$ implies $g(x,y) \subset D(y)$;
(ii)for all $y \in X$, $g(\cdot, y)$ is lower semicontinuous and $D(y)$ is closed;
(iii) for all $x \in X$, $f(x, \beta v + (1-\beta)w) \subset \beta f(x,v) + (1-\beta)f(x,w)$ for all $v, w \in X$

and $\beta \in [0,1]$ and $C(x)$ is a convex set;
(iv) for all $x \in X$, if $g(x,y) \subset D(y)$ for all $y \in X$, then $f(x,y) \not\subset C(x)$ for all $y \in X$;
(v) for all $x \in X$, $f(x,x) \not\subset C(x)$; and
(vi)there exists a nonempty compact subset K of X such that for all $M \in \langle X \rangle$, there exists a compact convex subset L_M of X containing M such that

$$L_M \cap \bigcap \{X \setminus intF^-(y) : y \in L_M\} \subset K,$$

where $F : X \multimap X$ is defined by $F(x) = \{y \in X : f(x,y) \subset C(x)\}$.
Then there exists an $\overline{x} \in X$ such that $f(\overline{x}, y) \not\subset C(\overline{x})$ for all $y \in X$.

The following Theorem follows from Lemma 1 and Theorem 2.

As a consequence of theorems 3 and 4, we have the following two theorems of equilibria for fuzzy maps.

Theorem 4. Let X be a convex subset of a topological vector space E, Z be a real topological vector space. $c, d : X \to \mathcal{F}(Z)$; $F, G : X \times X \to \mathcal{F}(Z)$; $\alpha : X \to (0,1]$ satisfy the following conditions:
(i)for all $x, y \in X$, if there exists z such that $F_{x,y}(z) \geq \alpha(x)$ and $c_x(z) < \alpha(x)$, then for all z with $G_{x,y}(z) \geq \alpha(y)$ implies $d_y(z) \geq \alpha(y)$;
(ii)for all $y \in X$, $G(\cdot, y)$ is topologically open and d is closed;
(iii)for all $x \in X$, $(F_{x,\beta v+(1-\beta)w})_{\alpha(x)} \subset \beta(F_{x,v})_{\alpha(x)} + (1-\beta)(F_{x,w})_{\alpha(x)}$ for all $v, w \in X$ and $\beta \in [0,1]$ and c is a convex fuzzy mapping;
(iv)for all $x \in X$, if for all z with $G_{x,y}(z) \geq \alpha(y)$ implies $d_y(z) \geq \alpha(y)$ for all $y \in X$, then for all $y \in X$, there exists a z such that $F_{x,y}(z) \geq \alpha(x)$ and $c_x(z) < \alpha(x)$;
(v)for all $x \in X$, there exists a z such that $F_{x,x}(z) \geq \alpha(x)$ and $c_x(z) < \alpha(x)$; and
(vi)there exists a nonempty compact subset K of X such that for all $M \in \langle X \rangle$ there exists a compact convex subset L_M of X containing M such that

$$L_M \cap \bigcap \{X \setminus intA(y) : y \in L_M\} \subset K,$$

where $A(y) = \{x \in X : \text{for all } z \text{ with } F_{x,y}(z) \geq \alpha(x) \text{ implies } c_x(z) \geq \alpha(x)\}$.
Then there exists an $\overline{x} \in X$ such that $(F_{\overline{x},y})_{\alpha(\overline{x})} \not\subset (c_{\overline{x}})_{\alpha(\overline{x})}$ for all $y \in X$.

From[7], we see that, condition (iv) in Theorem 3 can be replaced by:
(i)for all $x, y \in X$ with $y \neq x$ and $u \in]x, y[$, if $g(x,u) \subset D(u)$ and $f(u,y) \subset C(u)$, then $f(u,v) \subset C(u)$ for all $v \in]x, y[$.
(ii)for all $x, y \in X$ with $y \neq x$, the set $\{u \in [x,y] : f(u,y) \subset C(u)\}$ is open in $[x,y]$.
Hence we have:

Theorem 5. Let X be a convex subset of a topological vector space E, Z be a real topological vector space and $C, D : X \multimap Z$; $f, g : X \times X \multimap Z$ satisfy the following conditions:
(i)for all $x, y \in X$, $f(x,y) \not\subset C(x)$ implies $g(x,y) \subset D(y)$;
(ii)for all $y \in X$, $g(\cdot, y)$ is lower semicontinuous and $D(y)$ is closed;
(iii) for all $x \in X$, $f(x, \beta v + (1-\beta)w) \subset \beta f(x,v) + (1-\beta) f(x,w)$ for all $v, w \in X$ and $\beta \in [0,1]$ and $C(x)$ is convex;
(iv)for all $x, y \in X$ with $y \neq x$ and $u \in]x, y[$, if $g(x,u) \subset D(u)$ and $f(u,y) \subset C(u)$,

then $f(u,v) \subset C(u)$ for all $v \in]x,y[$;
(v)for all $x, y \in X$ with $y \neq x$, the set $\{u \in [x,y] : f(u,y) \subset C(u)\}$ is open in $[x,y]$;
(vi) for all $x \in X$, $f(x,x) \not\subset C(x)$; and
(vii)there exists a nonempty compact subset K of X such that for all $M \in \langle X \rangle$, there exists a compact convex subset L_M of X containing M such that

$$L_M \cap \bigcap \{X \setminus intF^-(y) : y \in L_M\} \subset K,$$

where $F : X \multimap X$ is defined by $F(x) = \{y \in X : f(x,y) \subset C(x)\}$.
Then there exists an $\overline{x} \in X$ such that $f(\overline{x}, y) \not\subset C(\overline{x})$ for all $y \in X$.

From[8], we see that, condition (v) in Theorem 7 can be replaced by:
(v′)for all $x, y \in X$ with $y \neq x$, $f(\cdot, y)$ is upper smeicontinuous with compact values on $[x,y]$ and graph C is open.
Hence we have:

Theorem 6. Let X be a convex subset of a topological vector space E, Z be a real topological vector space and $C, D : X \multimap Z$; $f, g : X \times X \multimap Z$ satisfy the following conditions:
(i)for all $x, y \in X$, $f(x,y) \not\subset C(x)$ implies $g(x,y) \subset D(y)$;
(ii)for all $y \in X$, $g(\cdot, y)$ is lower semicontinuous and $D(y)$ is closed;
(iii) for all $x \in X$, $f(x, \beta v + (1-\beta)w) \subset \beta f(x,v) + (1-\beta) f(x,w)$ for all $v, w \in X$ and $\beta \in [0,1]$ and $C(x)$ is convex;
(iv)for all $x, y \in X$ with $y \neq x$ and $u \in]x,y[$, if $g(x,u) \subset D(u)$ and $f(u,y) \subset C(u)$, then $f(u,v) \subset C(u)$ for all $v \in]x,y[$;
(v)for all $x, y \in X$ with $y \neq x$, $f(\cdot, y)$ is upper smeicontinuous with compact values on $[x,y]$ and graph C is open;
(vi) for all $x \in X$, $f(x,x) \not\subset C(x)$; and
(vii)there exists a nonempty compact subset K of X such that for all $M \in \langle X \rangle$, there exists a compact convex subset L_M of X containing M such that

$$L_M \cap \bigcap \{X \setminus intF^-(y) : y \in L_M\} \subset K,$$

where $F : X \multimap X$ is defined by $F(x) = \{y \in X : f(x,y) \subset C(x)\}$.
Then there exists an $\overline{x} \in X$ such that $f(\overline{x}, y) \not\subset C(\overline{x})$ for all $y \in X$.

In Chang and Zhu[3], they give the conditions that the multimap defined by the $\alpha(x)$-cut set of a fuzzy map has closed graph. In the following lemma, we give the conditions which ensure the multimap defined by the strong $\alpha(x)$-cut set of a fuzzy map has open graph.

By Theorem 8 and Lemma 3, we have the following equilibrium theorem for fuzzy maps.

Theorem 7. Let X be a convex subset of a topological vector space E, Z be a real topological vector space.$\alpha : X \to (0,1]$ be upper semicontinuous; $c, d : X \to \mathcal{F}(Z)$; $F, G : X \times X \to \mathcal{F}(Z)$ satisfy the following conditions
(i)for all $x, y \in X$, if there exists a $z \in Z$ such that $F_{x,y}(z) \geq \alpha(y)$ and $c_x(z) \leq \alpha(x)$, then for all z with $G_{x,y}(z) \geq \alpha(y)$ implies $d_y(z) \geq \alpha(y)$;
(ii)for all $y \in X$, $G(\cdot, y)$ is topologically open and d is a closed fuzzy mapping;

(iii)for all $x \in X$, $(F_{x,\beta v+(1-\beta)w})_{\alpha(\beta v+(1-\beta)w)} \subset \beta(F_{x,v})_{\alpha(v)} + (1-\beta)(F_{x,w})_{\alpha(w)}$ for all $v, w \in X$ and $\beta \in [0,1]$ and c is a convex fuzzy mapping;
(iv)for all $x, y \in X$ with $y \neq x$ and $u \in]x,y[$, if for all z with $G_{x,u}(z) \geq \alpha(u)$ implies $d_u(z) \geq \alpha(u)$ and for all z with $F_{u,y}(z) \geq \alpha(y)$ implies $c_u(z) > \alpha(u)$, then for all z with $F_{u,v}(z) \geq \alpha(v)$ implies $c_u(z) > \alpha(u)$ for all $v \in]x,y[$;
(v)for all $x, y \in X$ with $y \neq x$, $F(\cdot, y)$ is topologically closed on $[x,y]$; for all $x, y \in X$, $F_{x,y}$ is compact and $c_x(z)$ is lower semicontinuous on $X \times Z$ as real ordinary function;
(vi)for all $x \in X$, there exists a $z \in Z$ such that $F_{x,x}(z) \geq \alpha(x)$ and $c_x(z) \leq \alpha(x)$; and
(vii)there exists a nonempty compact subset K of X such that for all $M \in \langle X \rangle$ there exists a compact convex subset L_M of X containing M such that

$$L_M \cap \bigcap \{X \setminus intA(y) : y \in L_M\} \subset K,$$

where $A(y) = \{x \in X : \text{for all } z \text{ with } F_{x,y}(z) \geq \alpha(y) \text{ implies } c_x(z) > \alpha(x)\}$.
Then there exists an $\overline{x} \in X$ such that $(F_{\overline{x},y})_{\alpha(y)} \not\subset (c_{\overline{x}})^{\alpha(\overline{x})}$ for all $y \in X$.

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