Trends in Mathematics
Information Center for Mathematical Sciences
Volume 2, September 1999, Pages 1–

# ALGEBRAIC AND NASH STRUCTURES OF $C^\infty G$ MANIFOLDS, AND DEFINABLE $G$ MANIFOLDS IN AN $O$-MINIMAL EXPANSION OF $(\mathbb{R}, +, \cdot, <)$

TOMOHIRO KAWAKAMI

Abstract. The purpose of this article is an overview of results obtained as a part of a program to develop transformation groups in the algebraic, Nash, and definable category. The emphasis is on the algebraic and Nash realization of equivariant manifolds, and on the equivariant definable manifolds in $o$-minimal expansion of $(\mathbb{R}, +, \cdot, <)$. Finally, we make some questions related to the subject.

## 1. Introduction.

H. Seifert [16] proved that every closed stably paralellizable $C^\infty$ manifold is $C^\infty$ diffeomorphic to a union of connected components of a nonsingular algebraic set. J. Nash [13] posed the following conjecture in 1952.

*Nash conjecture.* Every closed $C^\infty$ manifold is $C^\infty$ diffeomorphic to a nonsingular algebraic set.

A. Tognoli [21] solved this conjecture affirmatively in 1973, and J. Bochnak and W. Kucharz [1] proved a stronger version of it which states that any closed $C^\infty$ manifold of positive dimension admits a continuous family of structures of nonsingular algebraic sets. A. Tognoli's proof consists of two parts. One is a cobordism process and the other is an approximation process. Using only the latter process, one can prove that every $C^\infty$ manifold is $C^\infty$ diffeomorphic to a union of connected components of some nonsingular algebraic set. The first process is required to get rid of its extra connected components.

K.H. Dovermann, M. Masuda and T. Petrie [3] conjectured the following in 1990.

*Equivariant Nash conjecture.* Let $G$ be a compact Lie group. Every closed $C^\infty G$ manifold is $C^\infty G$ diffeomorphic to a nonsingular algebraic $G$ set. Here a nonsingular algebraic $G$ set means a $G$ invariant nonsingular algebraic set in some representation of $G$.

They proved some partial solutions, in particular, for every closed $C^\infty G$ manifold $X$, the disjoint union $X \coprod X$ is $C^\infty G$ diffeomorphic to some nonsingular algebraic $G$ set [3]. In the equivariant setting, the approximation process works, but we do not know whether the cobordism process works or not. The main difficulty to obtain algebraic structures of $C^\infty G$ manifolds is to remove their extra connected components. The equivariant Nash conjecture remains open.

1991 *Mathematics Subject Classification*: 14P10, 14P15, 14P20, 57S05, 57S15, 58A07, 03C.
Key Words and Phrases: $C^\infty G$ manifolds, Nash $G$ manifolds, definable $C^\omega G$ manifolds, $o$-minimal, nonsingular algebraic sets.

Nash manifolds have good properties (cf. [17], [18], [19]) and they share some such properties with nonsingular algebraic sets. For example, a Nash manifold has only finitely many connected components, and it is either compact or compactifiable. Further one can consider definable $C^\omega$ manifolds in an $o$-minimal structure $M$ expanding $(\mathbb{R}, +, \cdot, <)$ [7], which are generalization of Nash manifolds. Definable $C^\omega$ manifolds have good properties [7] and many such $o$-minimal structures exist [22]. Definable $C^\omega G$ manifolds are generalizations of Nash $G$ manifolds, and fundamental results on them are stated in [6].

In this paper, every manifold does not have boundary unless otherwise stated.

Let $G$ be an affine Nash group. In general, the following implications hold:

a nonsingular algebraic $G$ set $\Longrightarrow$ an affine Nash $G$ manifold

$\Longrightarrow \begin{cases} \text{a Nash } G \text{ manifold} \\ \text{an affine definable } C^\omega G \text{ manifold} \end{cases} \Longrightarrow$ a definable $C^\omega G$ manifold

$\Longrightarrow$ a $C^\infty G$ manifold.

The following two problems are a compactifiable version of [21] and a relative version of [3].

*Problem 1.* Does a compactifiable $C^\infty$ manifold admit a structure of nonsingular algebraic set ? How many such structures does it admit ? Here a compactifiable $C^\infty$ manifold means a noncompact $C^\infty$ manifold which is $C^\infty$ diffeomorphic to the interior of some compact $C^\infty$ manifold with boundary.

*Problem 2.* Let $G$ be a compact Lie group. Let $X$ be a closed $C^\infty G$ manifold and let $X_1, \dots, X_n$ be closed $C^\infty G$ submanifolds of $X$. Under what condition, does $(X; X_1, \dots, X_n)$ admit a simultaneous algebraic structure ?

The following is a problem on Nash structures of $C^\infty G$ manifolds.

*Problem 3.* Let $G$ be a compact affine Nash group and let $X$ be a $C^\infty G$ manifold. Suppose that $X_1, \dots, X_n$ are $C^\infty G$ submanifolds of $X$.
(1) What is a necessary and sufficient condition for $X$ to be $C^\infty G$ diffeomorphic to an affine Nash $G$ manifold ?
(2) How many affine or nonaffine Nash structures does $X$ admits ?
(3) When does $(X; X_1, \dots, X_n)$ admit a simultaneous affine or nonaffine Nash structure ? How many such structures does it have ?

We consider the following problem because a $C^r$ $(r < \infty)$ Nash manifold is affine [17] and if $G$ is a compact Lie group, then every closed $C^\infty G$ manifold is $C^\infty G$ imbeddable into some representation of $G$ [14].

*Problem 4.* Let $r$ be a non-negative integer or $\infty$.
(1) Is every definable $C^r$ manifold affine ?
(2) Let $G$ be a compact definable $C^r$ group. Is every definable $C^r G$ manifold affine ?

## 2. Definable $G$ manifolds and Nash $G$ manifolds.

Let $M = (\mathbb{R}, +, \cdot, <, (f_i)_{i \in I}, (R_j)_{j \in J})$ be a structure on $(\mathbb{R}, +, \cdot, <)$, where $+$ (resp. $\cdot$) : $\mathbb{R}^2 \longrightarrow \mathbb{R}$ is the additive (resp. the multiplicative) function of $\mathbb{R}$, each $f_i : \mathbb{R}^{n(i)} \longrightarrow \mathbb{R}, n(i) \in \mathbb{N} \cup \{0\}$ is a function, and each $R_j \subset \mathbb{R}^{n(j)}, n(j) \in \mathbb{N}$ is a relation. Here we identify $f_i$ with an element of $\mathbb{R}$ if $n(i) = 0$. We say that $f$ (resp. $R$) is an $m$-place function symbol (resp. an $m$-place relation symbol) if $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ is a function (resp. $R \subset \mathbb{R}^m$ is a relation). Recall the definition of definable sets and $o$-minimal structures (cf. [2], [4]).

A *term* is a finite string of symbols obtained by repeated applications of the following two rules:
(a) Variables are terms.
(b) If $f$ is an $m$-place function symbol of $M$ and $t_1, \dots, t_m$ are terms, then the concatenated string $f(t_1, \dots, t_m)$ is a term.
Remark that if $m = 0$, then (b) says that constant symbols (0-place function symbols) are terms.

A *formula* is a finite string of symbols $s_1 \dots s_k$, where each $s_i$ is either a variable, a function symbol, a relation symbol, one of the logical symbols $=, \neg, \vee, \wedge, \exists, \forall$, one of the brackets (, ), or comma ,. Arbitrary formulas are generated inductively by the following three rules:
(a) For any two terms $t_1$ and $t_2$, $t_1 = t_2$ and $t_1 > t_2$ are formulas.
(b) If $R$ is an $m$-place relation symbol and $t_1, \dots, t_m$ are terms, then $R(t_1, \dots, t_m)$ is a formula.
(c) If $\phi$ and $\psi$ are formulas, then the negation $\neg\phi$, the disjunction $\phi \vee \psi$, and the conjunction $\phi \wedge \psi$ are formulas. If $\phi$ is a formula and $v$ is a variable, then $(\exists v)\phi$ and $(\forall v)\phi$ are formulas.

A subset $X$ of $\mathbb{R}^n$ is *definable* (in $M$) if it is defined by a formula (with parameters).

Let $K \subset \mathbb{R}^n, L \subset \mathbb{R}^m$ be definable sets. We say that a map $f : K \longrightarrow L$ is *definable* if the graph of $f$ $(\subset K \times L \subset \mathbb{R}^n \times \mathbb{R}^m)$ is definable. Let $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$ be open definable sets and let $r$ be a non-negative integer, $\infty$ or $\omega$. A $C^r$ map $f : U \longrightarrow V$ is called a *definable $C^r$ map* if it is definable. A definable $C^r$ map $h : U \longrightarrow V$ is called a *definable $C^r$ diffeomorphism* (a *definable homeomorphism* if $r = 0$) if there exists a definable $C^r$ map $k : V \longrightarrow U$ such that $h \circ k = id$ and $k \circ h = id$.

An *open interval* means something of the form $(a, b), a, b \in \mathbb{R} \cup \{\pm\infty\}$. We call $M$ *o-minimal* if every definable subset of $\mathbb{R}$ is a finite union of points and open intervals.

We can consider the following two definitions without assuming $o$-minimality of $M$. But if $M$ is not $o$-minimal, then the things defined in the following do not have satisfactory properties.

*Definition 2.1.* Let $r$ be a non-negative integer, $\infty$ or $\omega$.
(1) A definable subset $X$ of $\mathbb{R}^n$ is called a *definable $C^r$ submanifold of dimension $d$* if for any $x \in X$ there exists a definable $C^r$ diffeomorphism (a definable homeomorphism if $r = 0$) $\phi$ from some open definable neighborhood $U$ of the origin in $\mathbb{R}^n$ onto some open definable neighborhood $V$ of $x$ in $\mathbb{R}^n$ such that $\phi(0) = x, \phi(\mathbb{R}^d \cap U) = X \cap V$. Here $\mathbb{R}^d$ denotes the subset of $\mathbb{R}^n$ those which the last $(n - d)$ components are zero.
(2) A *definable $C^r$ manifold $X$ of dimension $d$* is a $C^r$ manifold with a finite system of charts $\{\phi_i : U_i \longrightarrow \mathbb{R}^d\}$ such that for each $i$ and $j$, $\phi_i(U_i \cap U_j)$ is an open definable subset of $\mathbb{R}^d$ and the map $\phi_j \circ \phi_i^{-1} | \phi_i(U_i \cap U_j) : \phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j)$ is a definable $C^r$ diffeomorphism (a definable homeomorphism if $r = 0$). We call these charts of class *definable $C^r$*. Definable $C^r$ manifolds with compatible atlases are identified. A subset $Y$ of $X$ is called a *$k$-dimensional definable $C^r$ submanifold* of $X$ if each point $x \in Y$ there exists a chart $\phi_i : U_i \longrightarrow \mathbb{R}^d$ of $X$ such that $x \in U_i$, $\phi_i(x) = 0$ and $U_i \cap Y = \phi_i^{-1}(\mathbb{R}^k)$, where $\mathbb{R}^k \subset \mathbb{R}^d$ is the vectors whose last $(d - k)$

components are zero. By [7], if $r > 0$, then a definable $C^r$ submanifold of $\mathbb{R}^n$ admits a finite family of definable $C^r$ charts, thus it is of course a definable $C^r$ manifold.
(3) Let $X$ (resp. $Y$) be a definable $C^r$ manifold with definable $C^r$ charts $\{\phi_i : U_i \longrightarrow \mathbb{R}^n\}_i$ (resp. $\{\psi_j : V_j \longrightarrow \mathbb{R}^m\}_j$). A $C^r$ map $f : X \longrightarrow Y$ is said to be a *definable $C^r$ map* if for any $i$ and $j$ $\phi_i(f^{-1}(V_j) \cap U_i)$ is open and definable in $\mathbb{R}^n$ and the map $\psi_j \circ f \circ \phi_i^{-1} : \phi_i(f^{-1}(V_j) \cap U_i) \longrightarrow \mathbb{R}^m$ is a definable $C^r$ map.
(4) Let $X$ and $Y$ be definable $C^r$ manifolds. We say that $X$ is *definably $C^r$ diffeomorphic to* $Y$ if one can find definable maps $f : X \longrightarrow Y$ and $h : Y \longrightarrow X$ such that $f \circ h = id$ and $h \circ f = id$.
(5) A definable $C^r$ manifold is said to be *affine* if it is definably $C^r$ diffeomorphic to a definable $C^r$ submanifold in some Euclidean space $\mathbb{R}^l$.

*Definition 2.2.* Let $r$ be a non-negative integer, $\infty$ or $\omega$.
(1) A group $G$ is called a *definable $C^r$ group* (resp. an *affine definable $C^r$ group*) if $G$ is a definable $C^r$ manifold (resp. an affine definable $C^r$ manifold) and that the multiplication $G \times G \longrightarrow G$ and the inversion $G \longrightarrow G$ are definable $C^r$ maps.

Let $G$ be a definable $C^r$ group.
(2) A definable $C^r$ submanifold in a representation of $G$ is called a *definable $C^rG$ submanifold* if it is $G$ invariant.
(3) A *definable $C^rG$ manifold* is a pair $(X, \theta)$ consisting of a definable $C^r$ manifold $X$ and a group action $\theta$ of $G$ on $X$ such that $\theta : G \times X \longrightarrow X$ is a definable $C^r$ map. For simplicity of notation, we write $X$ instead of $(X, \theta)$.

One can consider definable $C^rG$ maps, definable $C^rG$ diffeomorphisms, and affine definable $C^rG$ manifolds in the similar way.

If $M = (\mathbb{R}, +, \cdot, <)$, then a definable subset of $\mathbb{R}^n$ is a semialgebraic subset of it [20]. A definable $C^\omega$ manifold (resp. definable $C^\omega$ group, definable $C^\omega G$ manifold) in $(\mathbb{R}, +, \cdot, <)$ is called a *Nash manifold* (resp. a *Nash group*, a *Nash $G$ manifold*). We say that an affine definable $C^\omega$ group (resp. an affine definable $C^\omega G$ manifold ) in $(\mathbb{R}, +, \cdot, <)$ is an *affine Nash group* (resp. an *affine Nash $G$ manifold*).

## 3. Results.

**Theorem 3.1 [9].** *Every compactifiable $C^\infty$ manifold $X$ of positive dimension admits infinitely many nonsingular algebraic sets $\{Y_n\}_{n \in \mathbb{N}}$ of some $\mathbb{R}^k$ such that each $Y_n$ is $C^\infty$ diffeomorphic to $X$ and that $Y_n$ is birationally inequivalent to $Y_m$ for $n \neq m$.*

Let $G$ be a compact Lie group, $X$ a closed $C^\infty G$ manifold, and $X_1, \dots, X_n$ closed $C^\infty G$ submanifolds of $X$ in general position. Here $X_1, \dots, X_n$ are in general position (in $X$) means that for each $i \in \{1, \dots, n\}$ and $J \in \{1, \dots, n\} - \{i\}$, $X_i$ intersects transverse to $\cap_{i \in J} X_j$. We say that $(X; X_1, \dots, X_n)$ is *simultaneously algebraically $G$ cobordant* if there exist a nonsingular algebraic $G$ set $Y$, nonsingular algebraic $G$ subsets $Y_1, \dots, Y_n$ of $Y$, a $G$ cobordism $N$ between $X$ and $Y$, and $G$ cobordisms $N_i$ $(1 \leq i \leq n)$ between $X_i$ and $Y_i$ such that each $N_i$ is a $C^\infty G$ submanifold of $N$ and $N_1, \dots, N_n$ are in general position.

**Theorem 3.2 [10].** *Let $G$ be a compact Lie group and let $X$ be a closed $C^\infty G$ manifold. Suppose that $X_1, \dots, X_n$ are closed $C^\infty G$ submanifolds of $X$ in general position. If $(X; X_1 \dots, X_n)$ is simultaneously algebraically $G$ cobordant, then*

*$(X; X_1, \dots, X_n)$ is simultaneously $C^\infty G$ diffeomorphic to a tuple $(Y; Y_1, \dots, Y_n)$ of a nonsingular algebraic $G$ set and its nonsingular algebraic $G$ subsets. In particular, for any closed $C^\infty G$ manifold $Z$ and its closed $C^\infty G$ submanifolds $Z_1, \dots, Z_n$ of $Z$ in general position, a tuple $(Z \amalg Z; Z_1 \amalg Z_1, \dots, Z_n \amalg Z_n)$ of disjoint unions admits a simultaneous algebraic structure.*

A $C^\infty G$ manifold is *compactifiable* if it is noncompact and $C^\infty G$ diffeomorphic to the interior of some compact $C^\infty G$ manifold with boundary.

**Theorem 3.3** [8]. *Let $G$ be a compact affine Nash group and let $X$ be a $C^\infty G$ manifold.*
*(1) $X$ admits an affine Nash manifold structure if and only if $X$ is either compact or compactifiable.*
*(2) If $X$ is compact, then its affine Nash manifold structure is unique up to Nash $G$ diffeomorphism.*
*(3) If $X$ is either compact, connected, of positive dimension, and with non-transitive action, or compactifiable, then $X$ admits a uncountable family of nonaffine Nash $G$ manifold structures.*

Let $X$ be a noncompact $C^\infty G$ manifold and let $X_1, \dots, X_n$ be noncompact $C^\infty G$ submanifolds of $X$ in general position. We call $(X; X_1, \dots, X_n)$ *simultaneously compactifiable* if there exist a compact $C^\infty G$ manifold $Y$ with boundary $\partial Y$, compact $C^\infty G$ submanifolds $Y_i$ of $Y$ with boundary $\partial Y_i$ $(1 \le i \le n)$, and a $C^\infty G$ diffeomorphism $f : X \longrightarrow \operatorname{Int} Y$ such that $f(X_i) = \operatorname{Int} Y_i, \partial Y_i \subset \partial Y$ $(1 \le i \le n)$, and $Y_1, \dots, Y_n$ and $\partial Y$ are in general position.

**Theorem 3.4** [11]. *Let $G$ be a compact affine Nash group. Suppose that $X$ is a noncompact $C^\infty G$ manifold and $X_1, \dots, X_n$ are noncompact $C^\infty G$ submanifolds in general position such that no $X_i$ is contained in a compact subset of $X$. Then $(X; X_1, \dots, X_n)$ is simultaneously compactifiable if and only if it admits a simultaneous affine Nash $G$ manifold structure.*

Let $(X; X_1)$ and $(Y; Y_1)$ be two pairs of Nash manifolds and its Nash submanifolds. We say that they are *weakly simultaneously Nash diffeomorphic* if there exists a simultaneous $C^\infty$ diffeomorphism $f : (X; X_1) \longrightarrow (Y; Y_1)$ such that $f|X_1 : X_1 \longrightarrow Y_1$ is a Nash diffeomorphism.

**Proposition 3.5** [11]. *Let $X$ be an affine Nash manifold and let $X_1$ be a Nash submanifold of $X$ with $\dim X_1 \ge 1$. Then $(X; X_1)$ admits uncountably many pairs of $(X^c; X_1^c)_{c \in \Lambda}$ of nonaffine Nash manifolds and its nonaffine Nash submanifolds such that each $(X^c; X_1^c)$ is simultaneously $C^\infty$ diffeomorphic to $(X; X_1)$, and that $(X^c; X_1^c)$ is not weakly simultaneously Nash diffeomorphic to $(X^{c'}; X_1^{c'})$ for $c \ne c'$.*

A structure $M$ on $(\mathbb{R}, +, \cdot, <)$ is *polynomially bounded* if for any function $f : \mathbb{R} \longrightarrow \mathbb{R}$ definable in $M$, there exist an integer $l$ and a real number $x_0$ such that $|f(x)| \le x^l$ for all $x > x_0$. Otherwise, $M$ is called *exponential*.

**Theorem 3.6** [7]. *Let $M$ be an o-minimal expansion of $(\mathbb{R}, +, \cdot, <)$.*
*(1) If $M$ is polynomially bounded and $r$ is a non-negative integer, then every definable $C^r$ manifold is affine.*
*(2) If $M$ is exponential, $G$ is a compact affine definable $C^\infty$ group, then every compact definable $C^\infty G$ manifold is affine.*

## 4. Sketches of proofs.

In this section we sketch proofs of our results.

*The simplest case of Theorem 3.1.* Let $X$ be the unit circle $\{(x,y)|x^2+y^2=1\} \subset \mathbb{R}^2 \subset \mathbf{P}^2(\mathbb{R})$. For $n \geq 2$, let $Y_n = \{(x,y)|x^{2n}+y^{2n}=1\} \subset \mathbb{R}^2 \subset \mathbf{P}^2(\mathbb{R})$. Then each $Y_n$ is $C^\infty$ diffeomorphic to $X$. Using birational invariance of arithmetic genus, $Y_n$ is birationally inequivalent to $Y_m$ for $n \neq m$.

In the general case, to distinguish $Y_n$, we use birational invariance of Hodge number. $\square$

Let $G$ be a Lie group and let $I=[0,1]$. Suppose that $X'$ is a $C^\infty G$ submanifold of a $C^\infty G$ manifold $X$ and $i : X' \longrightarrow X$ is the inclusion. A $G$ *isotopy* of $X'$ is a $C^\infty G$ map $h : X' \times I \longrightarrow X$ such that $h(x,0)=i(x)$ and $h_t := h|(X' \times \{t\}) : X' \times \{t\} \longrightarrow X$ is a $C^\infty G$ imbedding for each $t \in I$. A $G$ isotopy is said to be *small* if each $h_t$ is close to $i$.

**Proposition 4.1 [10].** *Let $G$ be a compact Lie group and let $\Omega$ be a representation of $G$. Suppose that $X$ is a $G$ invariant union of nonsingular connected components of an algebraic $G$ set in $\Omega$, and that $X'$ is a compact $C^\infty G$ submanifold of $X$. Then there exist a $G$ invariant union $Y' \subset X \times \Xi$ of compact nonsingular connected components of an algebraic $G$ set in $\Omega \times \Xi$ for some representation $\Xi$ of $G$ and an arbitrarily small $G$ isotopy which takes $X' \times \{0\}$ to $Y'$.*

Let $Z$ be a topological $G$ space and let $Y$ be a $G$ invariant subset of $Z$. Recall that $Y$ *separates $Z$ compactly and equivariantly* if there exist closed $G$ invariant subsets $Z_0$ and $Z_1$ of $Z$ such that $Z = Z_0 \cup Z_1$, $Y = Z_0 \cap Z_1$ and $Z_1$ is compact. Notice that $Y$ is compact.

For an algebraic set $X$, let $Nonsing\ X$ denote the set of nonsingular points of $X$

**Lemma 4.2 [10].** *Let $G$ be a compact Lie group. Let $Z$ be an algebraic $G$ set and let $Z_i$ $(1 \leq i \leq k)$ be algebraic $G$ subsets of $Z$. Suppose that $L \subset Z$ is a nonsingular algebraic $G$ set, and that $Y$ is a compact codimension one $G$ submanifold of $Nonsing\ Z$ which contains $L$ and separates $Z$ compactly and equivariantly. Suppose further that $Y_i \subset Y$ $(1 \leq i \leq k)$ are compact codimension one $G$ submanifolds of $Nonsing\ Z_i$ such that $Y_1, \dots, Y_k$ are in general position in $Y$ and that $Y \cap Z_i = Y_i$ $(1 \leq i \leq k)$. Then there exists an arbitrarily small $C^\infty G$ isotopy $h_t$ of $Nonsing\ Z$ fixing $L$ such that it takes $Y$ and $Y_i$ $(1 \leq i \leq k)$ to a nonsingular algebraic $G$ set $W$ and nonsingular algebraic $G$ subsets $W_i$ $(1 \leq i \leq k)$ of $W$, respectively.*

By induction on $n$, Theorem 3.2 follows from Proposition 4.1 and Lemma 4.2.

*Sketch of proof of Theorem 3.3.* (1) Using the proof of [15], a noncompact affine Nash $G$ manifold is compactifiable.

If $X$ is compact, then $X$ is $C^\infty G$ diffeomorphic to an affine Nash $G$ manifold by [3]. If $X$ is compactifiable, then $X$ is $C^\infty G$ diffeomorphic to the interior of compact $C^\infty G$ manifold $Y$ with boundary $\partial Y$. Considering the pair $(D, \partial Y)$, where $D$ is the double of $Y$, we can show that $X$ admits an affine Nash $G$ manifold structure.

(2) follows from the Weierstrass polynomial approximation theorem and existence of a Nash $G$ tubular neighborhood of an affine Nash $G$ manifold.

(3) By (1), we may assume that $X$ is an affine Nash $G$ manifold, and by assumption of (3), there exists a Nash $G$ tubular neighborhood $U$ of some orbit such that

$U \neq X$. Using $U$, we construct a $G$ invariant open covering $\{U_i\}_{i=1}^3$ of $X$. Pasting them with Nash $G$ diffeomorphisms $f_\lambda^i, \lambda \in \Lambda$, where $\Lambda$ is a uncountable set, we have a uncountable family $\{Y_\lambda\}_{\lambda\in\Lambda}$ of Nash $G$ manifolds. Showing that each $Y_\lambda$ is nonaffine and for any $\lambda \in \Lambda$, the set of $\mu$ such that $Y_\mu$ is Nash $G$ diffeomorphic to $Y_\lambda$ is at most countable, we have (3). □

*Sketch of proof of Theorem 3.4.* At first, we prove that if $Z$ is a closed $C^\infty G$ manifold and $Z_1, \dots, Z_n$ are closed $C^\infty G$ submanifolds of $Z$ in general position, then $(Z; Z_1, \dots, Z_n)$ admits a simultaneous affine Nash $G$ manifold structure.

If $(X; X_1, \dots, X_n)$ is simultaneously compactifiable, there exist a compact $C^\infty G$ manifold $Y$ with boundary $\partial Y$, compact $C^\infty G$ submanifolds $Y_i$ of $Y$ with boundary $\partial Y_i$ $(1 \leq i \leq n)$, and a $C^\infty G$ diffeomorphism $f : X \longrightarrow \text{Int } Y$ such that $f(X_i) = \text{Int } Y_i, \partial Y_i \subset \partial Y$ $(1 \leq i \leq n)$, and $Y_1, \dots, Y_n$ and $\partial Y$ are in general position. Hence we can find the simultaneous double $(D; D_1, \dots, D_n)$ of $(Y; Y_1, \dots, Y_n)$. Applying $(D; D_1, \dots, D_n, \partial Y)$ to the first case, we have a simultaneous Nash $G$ manifold structure $(W; W_1, \dots, W_n, W')$ of $(D; D_1, \dots, D_n, \partial Y)$. Thus an appropriate $G$ invariant union of connected components $(\hat{X}; \hat{X}_1, \dots, \hat{X}_n)$ of $(W - W'; W_1 - W', \dots, W_n - W')$ is the desired structure.

Appropriate modifications of the proof of Theorem 3 and Corollary 4 [15] prove the converse. □

Let $X$ be an affine Nash manifold with boundary $\partial X$ and let $X_1$ be a Nash submanifold of $X$ with boundary $\partial X_1$ with $\partial X_1 \subset \partial X$. A Nash imbedding $\phi : \partial X \times [0,1] \longrightarrow X$ is a *relative Nash collar* of $\partial X$ in $(X; X_1)$ if $\phi|\partial X \times 0$ is the inclusion $\partial X \longrightarrow X$ and $\phi(\partial X_1 \times [0,1]) = \phi(\partial X \times [0,1]) \cap X_1$.

Let $B_\epsilon = \{x \in \mathbb{R}^k \,|\, ||x|| \leq \epsilon\}, \epsilon > 0$ and let $\mathbb{R}^l \subset \mathbb{R}^k$ be vectors whose last $(k-l)$ components are 0. Notice that $\partial B_\epsilon$ has a relative Nash collar in $(B_\epsilon; B_\epsilon \cap \mathbb{R}^l)$.

*Sketch of proof of Proposition 3.5.* Take $x \in X_1$. By the definition of Nash submanifolds, the above notice, and since $\dim X_1 \geq 1$, there exists an open semialgebraic neighborhood $U$ of $x$ in $X$ such that $U \cap X_1 \neq X_1$ and that the boundary $\partial\overline{U}$ of the closure $\overline{U}$ of $U$ in $X$ has a relative Nash collar $\phi$ in $(\overline{U}; \overline{U} \cap X_1)$. Using $\phi$, we have the required family in the generalized way of the proof of Theorem 3.3 (3). □

Recall that a subset of $\mathbb{R}^n$ is *locally closed* if it is the intersection of an open set and a closed set.

**Lemma 4.3 [5].** *Let $X \subset \mathbb{R}^n$ is a locally closed definable set and let $f, g$ be continuous definable function on $X$ with $f^{-1}(0) \subset g^{-1}(0)$. If $M$ is polynomially bounded, then for any compact subset $K$ of $X$, there exist a positive integer $N$ and a positive constant $c$ such that $|g^N| \leq c|f|$ on $K$.*

*Sketch of proof of Theorem 3.6 (1).* Let $X$ be a definable $C^r$ manifold. By the definition of definable $C^r$ manifold, there exists a finite family of definable $C^r$ chart $\{\phi_i : U_i \longrightarrow \mathbb{R}^n\}_{i=1}^k$ of $X$, where $n = \dim X$. Using Lemma 4.3, each $\phi_i$ can be extensible to a definable $C^r$ map $\phi_i' : M \longrightarrow \mathbb{R}^n$. Hence we can construct the required imbedding from $\phi_1', \dots, \phi_k'$. □

Let $G$ be a compact affine definable $C^\infty$ group. Let $X$ be a definable $C^\infty G$ manifold and let $x \in X$. A $G_x$ invariant definable $C^\infty$ submanifold $S$ of $X$ is called a *linear definable $C^\infty$ slice* at $x$ in $X$ if there exist a representation $\Omega$ of $G_x$ and a definable $C^\infty G_x$ imbedding $j : \Omega \longrightarrow X$ such that $j(\Omega) = S$, $j(0) = x$, $GS$ is open in $X$, $S$ is affine as a definable $C^\infty G_x$ manifold, and

$$\mu : G \times_{G_x} S \longrightarrow GS \ (\subset X), [g, x] \mapsto gx$$

is a definable $C^\infty G$ diffeomorphism.

**Theorem 4.4 [7].** *Suppose that $G$ is a compact affine definable $C^\infty$ group, $X$ is a definable $C^\infty G$ manifold, and that $x \in X$. Then there exists a linear definable $C^\infty$ slice at $x$ in $X$.*

If an $o$-minimal expansion $M$ of $(\mathbb{R}, +, \cdot, <)$ is exponential, then by [12] the exponential function $\mathbb{R} \longrightarrow \mathbb{R}$ is definable in $M$. Using this fact, we have the following lemma.

**Lemma 4.5 [7].** *Let $M$ be an exponential $o$-minimal expansion of $(\mathbb{R}, +, \cdot, <)$ and let $G$ be a compact affine definable $C^\infty$ group. Suppose that $D_1$ and $D_2$ are open balls in a representation $\Omega$ of $G$ of radius $a$ and $b$ with same center the origin and $a < b$. If $A, B \in \mathbb{R}, A \neq B$, then there exists a $G$ invariant definable $C^\infty$ function $f$ on $\Omega$ such that $f = A$ on $D_1$ and $f = B$ on $\Omega - \overline{D_2}$.*

Theorem 3.6 (2) follows from Theorem 4.4 and Lemma 4.5.

## 5. Questions.

Let $G$ be a compact Lie group. Then a nonsingular algebraic $G$ set is either compact or compactifiable [15]. Hence the equivariant Nash conjecture can be described in the following form.

*Question 5.1.* Let $G$ be a compact Lie group. A $C^\infty G$ manifold is $C^\infty G$ diffeomorphic to a nonsingular algebraic $G$ set if and only if it is either compact or compactifiable.

The following is a relative version of the Nash conjecture without assuming the general position condition.

*Question 5.2.* Let $X$ be a closed $C^\infty$ manifold and let $X_1, \dots, X_n$ be closed $C^\infty$ submanifold of $X$. Under what condition, does there exist a $C^\infty$ diffeomorphism $f$ from $X$ to a nonsingular algebraic set $Y$ such that each $f(X_i)$ is a nonsingular algebraic subset of $Y$ ?

Let $M$ be an $o$-minimal expansion of $(\mathbb{R}, +, \cdot, <)$. If $M$ is exponential, then the definable $C^\infty$ function $f : \mathbb{R} \longrightarrow \mathbb{R}$ determined by

$$f(x) = \begin{cases} 0 & x \leq 0 \\ e^{-\frac{1}{x}} & x > 0 \end{cases}$$

is not analytic.

It is known that a $C^\infty$ Nash map between $C^\omega$ Nash manifolds is a $C^\omega$ one, and that $(\mathbb{R}, +, \cdot, <)$ is polynomially bounded.

*Question 5.3.* Let $M$ be an $o$-minimal expansion of $(\mathbb{R}, +, \cdot, <)$. Is it true that $M$ is polynomially bounded if and only if every definable $C^\infty$ map between open definable sets is analytic ?

## References


1. J. Bochnak and W. Kucharz, *Nonisomorphic algebraic models of a smooth manifold*, Math. Ann. **290** (1991), 1-2.
2. J. Denef and van den Dries, *$P$-adic and real subanalytic sets*, Ann. Math. **128** (1988), 79-138.
3. K.H. Dovermann, M. Masuda, and T. Petrie, *Fixed point free algebraic actions on varieties diffeomorphic to $\mathbb{R}^n$*, Progress in Math. 80, Birkhäuser (1990), 49-80.
4. van den Dries, *Tame topology and o-minimal structures*, London Math. Soc., Lecture Note Series 248, Cambridge University Press, 1998.
5. van den Dries and C. Miller, *Geometric category and o-minimal structure*, Duke Math. J. **84** (1996), 497-540.
6. T. Kawakami, *Equivariant differential topology on an o-minimal expansion of* $(\mathbb{R},+,\cdot,<)$, in preparation.
7. T. Kawakami, *Imbedding of manifolds defined on an o-minimal structures on* $(\mathbb{R},+,\cdot,<)$, Bull. Korean Math. Soc. **36** (1999), 183-201.
8. T. Kawakami, *Nash $G$ manifold structures of compact or compactifiable $C^\infty G$ manifolds*, J. Math. Soc. Japan **48** (1996), 321-331.
9. T. Kawakami, *Nonisomorphic algebraic models of Nash manifolds and compactifiable $C^\infty$ manifolds*, Osaka J. Math. **31** (1994), 831-835.
10. T. Kawakami, *Relative equivariant Nash conjecture on simultaneous algebraic $G$ cobordant closed $C^\infty G$ manifolds*, preprint.
11. T. Kawakami, *Simultaneous Nash structures of a compactifiable $C^\infty G$ manifold and its $C^\infty G$ submanifolds*, Bull. Fac. Edu. Wakayama Univ. Natur. Sci. **48** (1999), 1-20.
12. C. Miller, *Exponentiation is hard to avoid*, Proc. Amer. Math. Soc. **122** (1994), 257-259.
13. J. Nash, *Real algebraic manifolds*, Ann. of Math. **56** (1952), 405-421.
14. R. S. Palais, *Imbedding of compact differential transformation groups in orthogonal representations*, J. Math. Mech. **6** (1957), 673-678.
15. T. Petrie, *Smooth conjugacy of algebraic actions on affine varieties*, Topology **27** (1988), 473-477.
16. H. Seifert, *Algebarische approximation von mannigfalitigkeiten*, Math. Z. **41** (1936), 1-17.
17. M. Shiota, *Abstract Nash manifolds*, Proc. Amer. Math. Soc. **96** (1986), 155-162.
18. M. Shiota, *Approximation theorems for Nash mappings and Nash manifolds*, Trans. Amer. Math .Soc **293** (1986), 319-337.
19. M. Shiota, *Nash manifolds*, Lecture Note in Mathematics vol 1269, Springer-Verlag, 1987.
20. A. Tarski, *A decision method for elementary algebra and geometry* (1951), 2 nd edition. revised, Berkeley and Los Angeles.
21. A. Tognoli, *Su una congettura di Nash*, Annali Sc. Norm. Sup. Pisa **27** (1973), 167-185.
22. A Wilkie, *Model completeness results for expnasions of the real field by restricted Pfaffian functions and exponential functions*, J. Amer. Math. Soc. **9** (1996), 1051-1094.