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THESEIBERG-WITTEN INVARIANTS AND SYMPLECTIC 4-MANIFOLDS

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Abstract. Applying the Seiberg-Witten invariants , we have been able to discover new consequences for the toplogy of smooth 4-manifolds. Above all, symplectic manifolds are the most distinguished classes of 4-manifolds which have nonvanishing Seiberg-Witten invariant. We will dicuss couple of applications of SW invariants to symplectic 4-manifolds and some related unsolved problems

1. Introduction

Smooth 4-manifolds have been a subject of intense investigation since the 80's. There have been lots of questions related to the smooth structure of 4-manifolds, for instance whether any positive definite unimodular form can be realized as the intersection form of a simply connected smooth 4-manifold. In 1980, S.K. Donaldson answered the question negatively and proved that the only possible definite intersection form for a simply connected 4-manifold is the standard one. With the result of Freedman [DK], it can be shown that there is a topological 4-manifold having no smooth structures. This was shown on the way to understanding the solution space of a certain partial differential equation, the Anti-Self-Dual(ASD) equation on the smooth 4-manifold. Furthermore, Donaldson developed new invariants known as the Donaldson polynomials, which are polynomial invariants on the 2-dimensional homology of the manifold. The utility of these invariants was demonstrated for instance, by the construction of a counter-example to the smooth h-cobordism Theorem in dimension 4. The principal problem is to calculate these invariants. Generally, it is an a priori difficult problem since the ASD equations are hard to handle. There have been a number of successful attempts to compute the invariants using the relation between the ASD equations and stable bundles over algebraic surfaces. Algebro-geometric methods enable one to compute the Donaldson invariant for some cases, [DK],[FM1]. However, from the physicist's point of view, the Donaldson theory is known to be equivalent to a quantum field theory. In 1994, E. Witten published a paper that showed one can obtain the Donaldson invariants by counting the solutions of dual equations, which involve $U(1)$ gauge fields and spinors. He and Seiberg formulated a new set of equations, Seiberg-Witten equations, and defined new invariants, Seiberg-Witten invariants. Using these, Witten showed that the canonical line bundle of a minimal algebraic surface of general type is the only one characteristic class having nontrivial SW invariant up to sign,

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which means at least that the canonical line bundle is smooth invariant. Furthermore he conjectured the exact formula relating the Donaldson invariants and new ones. Whether the conjecture is true or not, this new invariant, now called the Seiberg-Witten invariant, can help one to reveal more structure of smooth 4-manifolds with rather simpler computations. Also the results which were proved by the Donaldson invariants have been reproved rather easily using this new one. Moreover some conjectures motivated by the study of the Donaldson polynomial invariants have been established using this Seiberg-Witten invariant. For instance the smooth invariance of the Kodaira dimension of complex surfaces and the Thom conjecture concerning curves in $\mathbf{C}P^2$ were established using this. Even more, C.H. Taubes made use of the Seiberg-Witten invariant to show that every symplectic 4-manifold has a nontrivial invariant and discovered a far-reaching relation between the Seiberg-Witten invariant and the Gromov invariants which count the number of pseudo-holomorphic curves for certain homology classes. Using this relation, he proved, for example, the uniqueness of the symplectic structure of $\mathbf{C}P^2$. In this article, we are going to make use of Seiberg-Witten invariants and apply them to 4-dimensional symplectic manifolds and discuss some related open problems.

2. The Seiberg-Witten equations and Invariants

The following setup for the Seiberg-Witten invariants can be found in [KM],[M], [T1],[W]. Let X be an oriented, Riemannian 4-manifold. The chosen metric on X defines a principal $SO(4)$ bundle, $Fr = P_{S0(4)} \to X$, of oriented, orthogonal frames in TX. A *spin*^{c} structure on X is a lift (or more properly, an equivalence class of lifts) of Fr to a principal $Spin^c(4)$ bundle $F \to X$. In this regard, one should note the identifications

- $Spin^c(4) = (SU(2) \times SU(2) \times U(1))/\{\pm 1\}$ and
- $SO(4) = (SU(2) \times SU(2))/\{\pm 1\}$. (2.1)

with the evident group homomorphism from the former to the latter which forgets the factor of $U(1)$ in the top of line above. We remark that there is at least one *spin*^{c} structure on every oriented 4-manifold. Moreover, the set $Spin^c$ of *spin*^{c} structures is naturally metric-independent and has the structure of a principal homogeneous space for the additive group $H^2(X;\mathbf{Z})$. With the preceding understood, fix a *spin*^{c} structure $F \to X$. Then F can be used to construct three useful associated vector bundles, S^+, S^- and L. The first two are associated via the representations $s_\pm : Spin^c(4) \to U(2) = (SU(2) \times U(1))/\{\pm 1\}$ which forget one or other factor of $SU(2)$ in the top line of (2.1). Thus, $S^\pm$ are $\mathbf{C}^2$ vector bundles over X with Hermitian metrics. Meanwhile, $L = \det(S^+) = \det(S^-)$ is associated to F via the representation of $Spin^c(4)$ on $U(1)$ which forgets both factors of $SU(2)$ in the first line of (1.1). (By way of comparison, the $\mathbf{R}^3$ bundles $\Lambda^+, \Lambda^- \to X$ of self-dual and anti-self dual 2-forms are associated to Fr via the representations of $SO(4)$ to $SO(3)$ which forget one or other factor of $SU(2)$ in the second line of (2.1).)

Note that the bundle $S^+ \oplus S^-$ is a module for the bundle of Clifford algebras of TX in the sense that there is an epimorphism $cl : TX \to Hom(S^+, S^-)$ which obeys $cl^* cl = -1$. The latter will be thought of as a homomorphism from $S^+ \otimes TX$ to S^-. Note that this homomorphism induces a homomorphism cl^+, from Λ^+ to $End(S^+)$.

The Seiberg-Witten equations: We consider that the Seiberg-Witten equations constitute a system of partial differential equations for a pair (A, ϕ), where A is a

Hermitian connection on the complex line bundle L and where ϕ is a section of S^+. These equations read:

- $D\!\!\!/_A \phi = 0$
- $F_A^+ = q(\phi) + i\mu$. (2.2)

In the first line above, $D\!\!\!/_A$ is the Dirac operator as defined using the connection A on L and the Levi-Civita connection on TX. Indeed, these two connections define a unique connection on S^+ and hence a covariant derivative ∇_A, which takes a section of S^+ and returns one of $S^+ \otimes T^*X$. With this understood, then $D\!\!\!/_A$ sends the section ϕ to the section $cl(\nabla_A \phi)$ of S^-. To be more explicit about $D\!\!\!/_A \phi$, let $\{e^i\}_{1 \le i \le 4}$ be an oriented orthonormal frame for TX. Then $D\!\!\!/_A \phi = \sum e_i \cdot \nabla^A_{e_i} \phi$ where $\cdot$ is the Clifford multiplication.

In the second line of (1.2), F_A is the curvature 2-form of the connection A on L; this is an imaginary valued 2-form. Then, F_A^+ is the projection of F_A onto $i\Lambda^+ \cong \Omega_2^+(X; i\mathbf{R})$. Meanwhile, $q(\cdot)$ is the quadratic map from S^+ to $i\Lambda^+$ which, up to a constant factor, sends $\eta \in S^+$ to the image of $\eta \otimes \eta^*$ under the adjoint map cl^+. Then

$$q(\eta) \quad = \quad \eta \otimes \eta^* - \frac{1}{2}(|\eta|^2) Id.$$

Finally, μ in the second line of (2.2) is any fixed, real-valued self-dual 2-form. (A different choice for μ, as with a different choice of Riemannian metric, will give a different set of equations.)

The Seiberg-Witten invariants: The Seiberg-Witten invariant for the given *spin*^{c} structure $\mathcal{L} \in Spin^c$ is obtained by making a suitable count of solutions of (2.2). We remark here that the group $C^\infty(X; S^1)$ acts on the space

of solutions of (2.2); a map φ sends (A, ϕ) to $(A - 2\varphi^{-1} d\varphi, \varphi\phi)$. Here, S^1 is thought of as the unit circle in $\mathbf{C}$. (The group acts freely at the solutions where ϕ is not identically zero.) The quotient space of solutions to (2.2) by $C^\infty(X; S^1)$ is called the moduli space and will be denoted by $\mathcal{M}(\mathcal{L}, g, \mu)$.

Here are some crucial facts about $\mathcal{M}_X(\mathcal{L}, g, \mu)$.

1. Let Q denote the cup product pairing of $H^2(X; \mathbf{R})$ and let $H^{2,+} \subset H^2(X; \mathbf{R})$ denote a maximum subspace on which Q is positive definite. Set $b_2^+ = \dim(H^{2+})$. Fix an orientation for the real line
   - $H^0(X; \mathbf{R}) \otimes \Lambda^{top} H^1(X; \mathbf{R}) \otimes \Lambda^{top} H^{2+}$. (2.3)

   This serves to orient $\mathcal{M}_X(\mathcal{L}, g, \mu)$.
2. When $b_2^+ \ge 1$, by the generic choice of metric or by μ, as long as $c_1(L)$ is rationally nonzero, $\mathcal{M}$ contains no reducible solutions, i.e. solutions with $\phi = 0$. Here generic means off of a set of codimension b_2^+.
3. The space $\mathcal{M}$ naturally has the structure of a real analytic variety. When $b_2^+ \ge 1$, the space $\mathcal{M}$ will be a smooth manifold for a generic choice of μ in (1.2). The dimension of this manifold is computed ,with the help of the Atiyah-Singer index theorem, to be
   - $d = \frac{1}{4}(c_1(L) \cdot c_1(L) - 2\chi(X) - 3\text{sign}(X))$. (2.4)

   Here, χ is the Euler characteristic of X and $\text{sign}(X)$ is the signature. Also, the notation $u \cdot v$ for classes $u, v \in H^2(X : \mathbf{Z})$ denotes the evaluation of their cup product on the fundamental class of X.
4. Fix a base point in X and let $C_0^\infty(X; S^1)$ denote the subset of maps which map said base point to 1. This is the group of based gauge transformations.

Let $\mathcal{M}^0(\mathcal{L}, g, \mu)$ be the moduli space of solutions to (2.2) modulo based gauge transformations. When $\mathcal{M}$ is a smooth manifold, the projection $\mathcal{M}^0 \to \mathcal{M}$ defines a principal S^1 bundle.

5. The space $\mathcal{M}$ is compact.

With the preceding understood, the Seiberg-Witten invariant is defined as follows:

Definition 2.1. *Let X be a compact, oriented 4-manifold with $b_2^+ \geq 1$ and let $\mathcal{L} \in Spin^c$ be a $spin^c$ structure on X. Choose an orientation for* (1.3). *The Seiberg-Witten invariant $SW(\mathcal{L})$ for $\mathcal{L}$ is defined as follows:*

a): *When $d < 0$ in* (1.4)*, the invariant is defined to be zero.*

b): *When $d = 0$ in* (1.4)*, choose μ in* (2.2) *to make $\mathcal{M}_X(\mathcal{L}, g, \mu)$ a smooth manifold. Then this $\mathcal{M}$ is a finite union of signed points and the Seiberg-Witten invariant is the sum over these points of the corresponding ± 1's (The signs come from the orientation in* (1.3)*.)*

c): *When $d > 0$ in* (1.4) *, also choose μ to make $\mathcal{M}$ a smooth manifold. This $\mathcal{M}$ is compact and oriented, so it has a fundamental class. The Seiberg-Witten invariant is obtained by pairing this fundamental class with the maximal cup product of the first Chern class of the line bundle $\mathcal{M}^0 \times_{S^1} \mathbf{C} \to \mathcal{M}$.*

Note that if the dimension d of $\mathcal{M}$ is an odd number, i.e. $b_2^+ + b_1$ is even, then the Seiberg-Witten invariants are zero.
If $b_2^+(X) > 1$ then, by the usual cobordism argument, one can show that $SW(\mathcal{L}, g, \mu)$ does not depend on the metric or the form μ. Hence if $b_2^+ > 1$ then the Seiberg-Witten invariants as presented in [M],[W] constitute a diffeomorphism invariant map $SW : Spin^c \to \mathbf{Z}$.

Chamber structure : In the case where $b_2^+ = 1$, there exists a chamber structure depending on the metric and perturbation term μ. Let $\Omega(X) = \{x \in H^2(X;\mathbf{R}) | x^2 = 1\}$, then $\Omega(X)$ has two components and an orientation of $H^+(X;\mathbf{R})$ determines the positive component, denoted $\Omega^+(X)$. For a given *spin*^{c} structure $\mathcal{L}$, the wall determined by $\mathcal{L}$, denoted by $W_{\mathcal{L}}$, is the set of $(\omega, t) \in \Omega^+(X) \times \mathbf{R}$ such that $2\pi\omega \cdot c_1(\mathcal{L}) + t = 0$. The *chamber determined by $\mathcal{L}$* is a connected component of $(\Omega^+(X) \times \mathbf{R}) - W_{\mathcal{L}}$. There is a map, the period map, from the space of metrics and perturbations to the space $\Omega^+(X) \times \mathbf{R}$ defined by taking g and μ to ω_g and $t = \int_X \omega_g \wedge \mu$ where ω_g is the unique harmonic, self-dual two form $\omega_g \in \Omega^+(X)$. The Seiberg-Witten invariant of $\mathcal{L}$ for g and μ is well-defined if the corresponding period point does not lie on a wall, and it depends on g and μ only through the chamber of the associated period point. The change of the invariant can be computed by the wall-crossing formula (see [KM],[LL]). Let $\mathcal{C}$ be a collection of *spin*^{c} structures, then a *common chamber* for $\mathcal{C}$ is a connected component of

$$\Omega^+(X) \times \mathbf{R} - \bigcup_{\mathcal{L} \in \mathcal{C}} W_{\mathcal{L}}.$$

Given a cohomology class $c \in H^2(X;\mathbf{Z})$ of negative square, a common chamber for $\mathcal{C}$ is called perpendicular to $\mathcal{C}$ if it contains a pair (ω, t) with ω perpendicular to c.

Involution map : There is a canonical involution on the set of *spin*^{c} structures

$$J : Spin^c(X) \to Spin^c(X)$$
$$c_1(J\mathcal{L}) = -c_1(\mathcal{L})$$

Moreover, this involution sets up an identification of moduli spaces

$$\mathcal{M}_X(\mathcal{L}, g, \mu) = \mathcal{M}_X(J\mathcal{L}, g, -\mu)$$

With the usual sign convention, we have

$$SW_X(\mathcal{L}, g, \mu) = (-1)^{\frac{1-b_1(X)+b_2^+(X)}{2}} SW_X(J\mathcal{L}, g, -\mu).$$

See [M]. When $b_2^+(X) = 1$, the involution induces an involution J on $\Omega^+(X) \times \mathbf{R}$ given by

$$J(\omega, t) = (\omega, -t).$$

3. 4-dimensional symplectic topology and some open problems

Via the classification of complex surfaces, it is well-known that every complex two dimensional manifold with even first betti number has a Kähler structure, which implies that there is a associated riemannian metric which locally looks like the complex plain with a standard metric up to 2nd order. The definition of symplectic manifold is a kind of natural generalization of Kählermanifolds without imposing any necessary integrable complex structure associated with it. Here is a definition of symplectic manifold,

Definition 3.1. (X, ω) *is a symplectic manifold if* ω *is a nondegenerate differential two form which is closed.*

The examples of nonKählerian symplectic manifold had not been known until late 70's by Thurston. After that, 4-dimensional topologist, R. Gomph [G] showed that every finitely presented group can be realized as fundamental group of a symplectic 4-manifold. His theorem implies that there are a number of symplectic 4-manifolds which cannot have any complex structure on it. In advent of Seiberg-Witten theory, C.H Taubes proved remarkable theorem which says that a symplectic 4-manifold has nonvanishing SW invariants and far-reaching relation between SW invariants and Gromov's.

Theorem 3.2. $(Taubes[T1, T2])$ *Let* (X, ω) *be a compact, closed symplectic 4-manifold with* $b_2^+(X) \geq 2$. *Let* $K_X^{-1} = c_1(TX)$ *be the first Chern class of the associated almost complex structure on* X.

- $SW_X(K_X^{-1}) = 1$
- $SW_X(K_X^{-1} + 2E) = Gr(E)$

In the theorem. the Gromov invariant, $Gr(E)$ is the "number" of pseudo-holomorphic curves which represent Poincare dual of E. Like the vanishing theorem for Donaldson Invariants, we can prove following.

Theorem 3.3. $(Witten[W])$ *Suppose* $X \cong X_1 \# X_2$ *with* $b_2^+(X_i) \geq 1$ *for each* i, *then* X *has vanishing Seiberg-Witten invariants for any spin*^{c} *structure.*

Connected sum problem in symplectic category: Combining above theorems, we can show that a symplectic 4-manifold can not be decomposed as a connected sum whose summand has a non-negative positive second betti number (b_2^+), so the existence of a symplectic structure on a connected sum of smooth four manifolds is quite restrictive. Suppose $X \# Y$ has a symplectic form, then one of the summands has negative definite standard intersection form and its fundamental group has no finite quotient [KMT]. Based on this, they conjecture the following

Conjecture 3.4. *Suppose the symplectic 4-manifold* $(X, \omega) \cong X_+ \# N$, *the negative definite part of the decomposition, say* N, *is simply connected, moreover it is diffeomorphic to a connected sum of* $\overline{\mathbf{C}P^2}$*'s.*

The conjecture says that "blow-up" process is the only possible way of constructing another symplectic 4-manifold by connecting sums. This conjecture is holds for some special cases , which says that if X is a ruled surface or its blow-up and $X\#N$ has a symplectic structure then $X\#N$ must be diffeomorphic to $X\#\overline{\mathbf{C}P^2}\#\cdots\#\overline{\mathbf{C}P^2}$. I We can prove the weak-version of this conjecture by showing that the only possible symplectic 4-manifolds which have vanishing SW invariant in some metric chamber must be ruled, $\mathbf{C}P^2$ or one of their blow-ups. We can actually classified the class of symplectic 4-manifolds which has vanishing SW invariants for some metric. Here is a classifying theorem.

Theorem 3.5. *Let (X,ω) be a 4-dimensional symplectic manifold. The following statements are equivalent.*

a) *X is $\mathbf{C}P^2$, ruled surfaces, or their blow-ups.*
b) *There is a symplectically embedded 2-sphere in X which has nonnegative self-intersection number.*
c) *X has a positive scalar curvature metric.*
d) *$SW_X(\mathcal{L},g)\equiv 0$for some metric g.*

Sketch of proof: *a)* $\Leftrightarrow$ *b)* By the work of D. McDuff and M. Gromov [Gr],[Mc], X must be ruled, $\mathbf{C}P^2$ or their blow-ups.
b) $\Rightarrow$ *c)* In $\mathbf{C}P^2$, the Fubini-Study metric has a positive scalar curvature. In other cases, for example, minimal ruled surface has a metric by taking round positive scalar curvature on the fiber $\cong S^2$, which has a positive scalar curvature. In general cases, suppose both X and Y have a positive scalar curvature metric then so does $X\#Y$ [GL]. Hence we can assure that $Y = X\#\overline{\mathbf{C}P^2}\#\cdots\#\overline{\mathbf{C}P^2}$ has a positive scalar curvature metric.
c) $\Rightarrow$ *d)* We can show by the following argument that there are no solutions for the Seiberg-Witten equations for any *spin*^{c} structure.

For the solution (A,ϕ) of the SW equations, let x be the maximum point of $|\phi|^2$. ¿From $\not{D}_A\phi = 0$ and the Weitzenbock formula for the Dirac operator [LM],

$$\begin{aligned} 0 &\leq \frac{1}{2}\triangle|\phi|^2(x) \\ &= \frac{1}{2}d^*d <\phi,\phi>= 2<\nabla_A^*\nabla_A\phi,\phi> -2<\nabla_A\phi,\nabla_A\phi> \\ &\leq <\not{D}_A^2\phi,\phi> - <\frac{s}{4}\phi,\phi> - <F_A^+\cdot\phi,\phi> \\ &\leq -\frac{s}{4}|\phi|^2(x) - \frac{1}{2}|\phi|^4(x) \end{aligned}$$

where s is the scalar curvature induced by the given metric.
So we always have a uniform bound for the spinor field, $|\phi|^2 \leq -\frac{s}{2}$. In our case, using the psc metric, the only possible solutions for the SW equations are reducible but the generic condition for the metric $F_A^+\cdot\omega_g \neq 0$ gives us that no such solutions exist [M],[W].
c) $\Rightarrow$ *d)* Let us briefly recall some background on symplectic manifolds. Let ω be a symplectic form on X, i.e. ω is a non-degenerate, closed differential two form. There is an almost complex structure J calibrating ω , i.e. $\omega(\cdot,J\cdot)$ defines a positive definite bilinear form on TX. Denote the metric induced by the symplectic form by g_ω. Note that the space of almost complex structures calibrating ω is contractible. There is a natural *spin*^{c} structure induced by J whose determinant line bundle is

(K_X^{-1}, J) where, $K_X^{-1} \cong \Lambda^2 TX_{\mathbf{C}}$. Let $c_1(K_X^{-1})$ be the first Chern class of the line bundle K_X^{-1} Once again, we may assume $b_2^+(X) = 1$. Let $\tilde{X}$ be a minimal model for X. Then $X = \tilde{X} \# n\overline{\mathbf{C}P^2}$ and let E be the exceptional class such that $E = \sum_{1 \le i \le n} E_i$ then $K_X = K_{\tilde{X}} - E$. By the assumption, $SW(K_X^{-1}, g_X) = 0$ for some metric g_X, by the blow-up formula $SW(K_{\tilde{X}}^{-1}, g_{\tilde{X}}) = 0$ Suppose that $K_{\tilde{X}}^2 \ge 0$, then there is no wall between the metrics $g_{\tilde{X}}$ and g_ω, i.e. $\omega_g \cdot K_{\tilde{X}}^{-1}$ always has the same sign as $\omega \cdot K_{\tilde{X}}^{-1}$,

$$SW(K_{\tilde{X}}^{-1}, g_{\tilde{X}}) = SW(K_{\tilde{X}}^{-1}, g_\omega) = 0.$$

Hence we have $c_1(K_{\tilde{X}}^{-1}) \cdot \omega > 0$, if not then $SW(K_{\tilde{X}}^{-1}, g_\omega) = SW_r(K_{\tilde{X}}^{-1}) = 0$ for sufficiently large r, which is impossible [T1],[T3]. So we have $c_1(K_{\tilde{X}}) \cdot \omega < 0$. We have $c_1(K_X)^2 < 0$ or $c_1(K_X) \cdot \omega < 0$. The following lemma completes this step.

Lemma 3.6. *Suppose X is a minimal symplectic manifold with $c_1(K_X)^2 < 0$ or $c_1(K_X) \cdot \omega < 0$. Then X contains a rational curve with nonnegative self-intersection number.*

Sketch of proof : Once again, by the theorem of Taubes [T2],[T3], if $b_2^+(X) > 1$ then $c_1(K_X)^2 \ge 0$ or $c_1(K_X) \cdot \omega \ge 0$, so we must have $b_2^+(X) = 1$. In this case, the general cobordism argument fails so that the Seiberg-Witten invariants do depend on the metric and perturbation. There is a chamber structure for any *spin*^{c} structure with possibly different SW invariants in each chamber. The change of the invariants from one chamber to the other is given by the wall-crossing formula [LL]. We are going to make use of the equivalence between the Seiberg-Witten and Gromov invariants which is established by C.H. Taubes [T3]. Eventually, we can show that there exists a number of cohomology classes E such that

$$Gr(E) = SW_r(K_X^{-1} + 2E) = SW_X(K_X^{-1} + 2E, g) + w.c.n(K_X^{-1} + 2E) \ne 0$$

where, w.c.n$(K_X^{-1} + 2E)$ is the wall-crosssing number and $SW_r(K_X^{-1} + 2E) = SW_X(g, -iF_{A_0}^+ - \frac{1}{4}r)$ is the Seiberg-Witten invariants with a large pertubation term. Among them, we can find some class has genus 0 ($g(E) = \frac{1}{2}(K_X \cdot E + E^2) + 1$) with nonnegative self-intersection number. This assure that there exists a embedded rational symplectic submanifold in it [J]. we can prove following theorem.

Theorem 3.7. *Let X be a ruled surface or $\mathbf{C}P^2$. If $X \# N$ has a symplectic structure then $X \# N$ must be ruled or $\mathbf{C}P^2$, or one of their blow-ups. It follows that N is simply connected negative definite, i.e. $\pi_1(N) = b_2^+(N) = 0$.*

Proof : We can generate a family of metrics on the connected sum $X \# N$ in which the "neck" is pinched down to zero radius. These Riemannian metrics can be formed by connecting the above psc metrics on $(X - D_t^4, g_X)$ and any metric on $(N - D_t^4, g_N)$ where t is the radius of ball D^4. Any *spin*^{c} structure c on $X \# N$ can be decomposed as $c_X + c_N$. For this family of metrics, it can be shown that a sequence of solutions to the Seiberg-Witten equations for c has a subsequence which converges away from the neck to solutions over the closed manifolds X and N corresponding to c_X and c_N respectively. However, by the choice of the psc (positive scalar curvature) metric g_X on X, there are no solutions for the ruled surface side. Hence using a connected sum metric $g_X \# g_N$ over a ball D_t^4 with small radius, we see that $X \# N$ has no solutions to the Seiberg-Witten equations.

Kähler structures on symplectic manifold: By the construction of R. Gomph, we can show that there are many symplectic 4-manifolds which do not support integrable complex structure i.e. they cannot have a Kähler structure. However, with this result, we can say that every symplectic 4-manifolds with vanishing SW invariants has a Kähler structure. This was shown by that there are a lots of homology classes which have nontrivial Gromov invariants, which is possible in the case $b_2^+(X) = 1$ i.e. ω is the unique harmonic self-dual two form. With this existence of symplectic submanifold for large number of homology classes, we may discuss the existence of Kähler structure on it.

Problem 3.8. *Let (X, ω) be a symplectic 4-manifold and $b_2^+(X) = 1$ and $b_1(X) = 0$ then Does X admit a Kähler structure on it?*

Simple type conjecture for the Seiberg-Witten Invariants: For the case of $b_2^+(X) > 1$, the known examples of smooth 4-manifolds which has nonvanishing Seiberg-Witten invariants has at least almost complex structure on it. That is, equivalently, to say that there are some *spin*^{c} structure $\mathcal{L}$ such that $SW_X(\mathcal{L}) \neq 0$ and the formal dimension of the Seiberg-Witten moduli space, $d = \frac{1}{4}(c_1(L) \cdot c_1(L) - 2\chi(X) - 3\text{sign}(X))$, is zero. This leads us a simple type conjecture. let us start with a definition. Let $\mathcal{L}$ be a Characteristic line bundle for X

Definition 3.9. *$\mathcal{L}$ is a Seiberg-Witten basic class if $SW_X(\mathcal{L}) \neq 0$*

Let $d(\mathcal{L}) = \frac{1}{4}(c_1(L) \cdot c_1(L) - 2\chi(X) - 3\text{sign}(X))$ be the formal dimension of Seiberg-Witten moduli space. We still have a open problem for the dimension of basic class.

Conjecture 3.10. *Suppose that X is a smooth 4-manifold with $b_2^+(X) > 1$, then every basic class for the Seiberg-Witten invariants has zero formal dimension.*

In general, we call above conjecture as a "simple type" conjecture for Seiberg-Witten invariants. The above conjecture is true for X has a symplectic structure [T2,T3].

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