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OPTIMAL CONTROL PROBLEMS IN FLUID MECHANICS

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ABSTRACT. The optimal control problems of incompressible, viscous fluid flows are considered. Both time dependent and time independent models for fluid flows are considered. We provide a model problem of flow control problem and discuss the existence of optimal solution and Lagrange multipliers. We also describe a gradient type numerical algorithm for solving the optimality system. Finally, we discuss a piecewise-in-time optimal control problem for optimization problems of time dependent Navier-Stokes equations.

1. INTRODUCTION

The control of fluid motions for the purpose of achieving some desired objective is crucial to many applications. Recently, considerable progress has been made in mathematical analysis and computation of controlling the viscous incompressible flows governed by the Navier-Stokes equations or related equations. Most of recent publications, *e.g.* [2],[5]-[12], [14] and [16]-[19], include one or more of the following components:

- the construction of mathematical models and the analysis of the mathematical models to answer questions about the existence and regularity of solutions and to derive necessary conditions that optimal controls and states must satisfy;
- the construction and analysis of discretization methods for determining approximate solutions of the optimal control problems, and the rigorous derivation of error estimates; and
- some iteration schemes for solving the system of partial differential equations, the development of computer code implementing discretization algorithms and also to solve problems of practical interest.

We restrict our attention to incompressible viscous flow. There are many articles in which compressible and/or inviscid flows. In §2, we provide a model problem of flow control problem and discuss the existence of optimal solution and Lagrange multipliers. In §3, We discuss a gradient type iteration algorithm for solving the optimality system. Then, in §4, we discuss a piecewise-in-time optimal control problem for optimization problems of time dependent Navier-Stokes equations.

1.1. Notation.

In order to make precise some of the formal discussions from this section, we introduce some function spaces and their norms, along with some related notations; for details see [1].

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For any nonnegative integer m, we define the Sobolev space $H^m(\Omega)$ by

$$H^m(\Omega) = \{p \in L^2(\Omega) \ : \ D^\alpha p \in L^2(\Omega) \text{ for } 0 \le |\alpha| \le m\},$$

where $D^\alpha p$ denotes the weak (or distributional) partial derivative and α is a muti-index, $|\alpha| = \sum_i \alpha_i$. Clearly, $H^0(\Omega) = L^2(\Omega)$. We equip $H^m(\Omega)$ with the norm

$$||p||_m = \Big\{ \sum_{0 \le |\alpha| \le m} ||D^\alpha p||_0^2 \Big\}^{\frac{1}{2}}.$$

The usual inner product associated with $H^m(\Omega)$ will be denoted by $(\cdot,\cdot)_m$. We will also make use of fractional order Sobolev spaces; see [1].

We will also make use of the subspaces

$$L_0^2(\Omega) := \{q \in L^2(\Omega) | \int_\Omega q d\Omega = 0\}$$

and

$$H_0^1(\Omega) = \Big\{\psi \in H^1(\Omega) \ : \ \psi = 0 \text{ on }, \Big\}.$$

For vector valued functions, we define the Sobolev space

$$\mathbf{H}^m(\Omega) := \{\mathbf{u} \ : \ u_i \in H^m(\Omega), i = 1, 2, 3 \},$$

where $\mathbf{u} = \{u_1, u_2, u_3\}$, and its associated norm by

$$||\mathbf{u}||_m = \Big\{ \sum_{i=1}^3 ||u_i||_m^2 \Big\}^{\frac{1}{2}}.$$

Of special interest will be the subspaces

$$\mathbf{H}_0^1(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega) \ : \ \mathbf{v}\Big|_{\partial\Omega} = 0\},$$

and

$$\mathbf{V} = \{\mathbf{v} \in \mathbf{H}_0^1(\Omega) \ : \ \nabla \cdot \mathbf{v} = 0\}.$$

Let $\mathbf{H}$ be the closer of $\mathbf{V}$ in $\mathbf{L}^2(\Omega)$. All subspaces are equipped with the norms inherited from the underlying spaces. For the spaces $L^2(\Omega)$ and $\mathbf{L}^2(\Omega)$, we denote the inner product by $(\cdot,\cdot)$ and the norm by $\|\cdot\|$ or $\|\cdot\|_0$.

2. Analysis of some control problems

We describe some model problems in control of fluid flows.

2.1. A model distributed control problem. We consider the incompressible Navier-Stokes equations in a smooth bounded two-dimensional domain Ω, on an interval of time $[0, T]$.

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in} \quad \Omega \times [0,T], \\ \nabla \cdot \mathbf{u} = 0 \quad \text{in} \quad \Omega \times [0,T], \\ \mathbf{u} = 0 \quad \text{on} \quad \partial\Omega \times [0,T], \\ \mathbf{u}|_{t=0} = \mathbf{u}_0. \end{cases} \tag{2.1}$$

Here $\mathbf{u} = (u_1, u_2)$ is the velocity vector, p pressure, $\nu > 0$ the kinematic viscosity; $\mathbf{f}$, which will be the control, represents volume forces.

We introduce the functional

$$\mathcal{J}(\mathbf{u},p,\mathbf{f}) = \frac{1}{2}\int_0^T \|\nabla\times\mathbf{u}(\mathbf{x},t)\|^2\,dt + \frac{\delta}{2}\int_0^T \|\mathbf{f}\|^2\,dt \tag{2.2}$$

The optimal control problem we consider is to seek a state variable $\mathbf{u}$ and a control $\mathbf{f}$ such that the functional (2.2) is minimized subject to (2.1). The real goal of optimization is to minimize the first term appearing in the definition (2.2). The functional (2.2) measures the vorticity of the flow. The control of vorticity has significant applications in science and engineering such as control of turbulence and control of crystal growth process. The second term in the cost functional (2.2) is added to limit the cost of control. The non-negative penalty parameter δ can be used to change the relative importance of the two terms appearing in the definition of the functional. The problem is then

$\mathcal{P}_1$**:** To minimize $\mathcal{J}(\mathbf{u},p,\mathbf{f})$, for $\mathbf{f}\in L^2(0,T;\mathbf{H})$.

Concerning the existence of an optimal control we have

Theorem 2.1. *For $\mathbf{u}_0$ given in $\mathbf{H}$, there exists at least an element $\hat{\mathbf{f}}$ in $L^2(0,T;\mathbf{H})$, and $\hat{\mathbf{u}}\in C([0,T];\mathbf{H})\cap L^2([0,T];\mathbf{V})$, such that $\mathcal{J}(\mathbf{u},p,\mathbf{f})$ attains its minimum at $\hat{\mathbf{f}}$ and $\hat{\mathbf{u}}=\hat{\mathbf{u}}_{\hat{\mathbf{f}}}$.*

The first-order necessary conditions of optimality can be obtained by the Fréchet derivatives of $\mathcal{J}$.

$$\mathcal{J}'(\mathbf{u},p,\hat{\mathbf{f}}) = 0, \tag{2.3}$$

or

$$\langle \mathcal{J}'(\mathbf{u},p,\hat{\mathbf{f}}),\mathbf{f}\rangle = 0, \quad \forall\mathbf{f}.$$

A convenient expression of $\mathcal{J}'$ can be given by using the adjoint state which is defined by the adjoint of the linearized equations. The adjoint equation is given by

$$\begin{cases} \dfrac{\partial\boldsymbol{\xi}}{\partial t} - \nu\Delta\boldsymbol{\xi} + (\nabla\hat{\mathbf{u}})^t\cdot\boldsymbol{\xi} - (\hat{\mathbf{u}}\cdot\nabla)\boldsymbol{\xi} + \nabla\phi = \nabla\times\nabla\times\hat{\mathbf{u}} \quad \text{in} \quad \Omega\times[0,T], \\ \nabla\cdot\boldsymbol{\xi} = 0 \quad \text{in} \quad \Omega\times[0,T], \\ \boldsymbol{\xi} = 0 \quad \text{on} \quad \partial\Omega\times[0,T], \\ \boldsymbol{\xi}(\mathbf{x},T) = 0, \quad \mathbf{x}\in\Omega. \end{cases} \tag{2.4}$$

Using $\boldsymbol{\xi} = \boldsymbol{\xi}(\nabla\times\nabla\times\hat{\mathbf{u}}_{\mathbf{f}})$ one can prove that

$$\mathcal{J}'(\mathbf{f}) = \delta\mathbf{f} + \boldsymbol{\xi}_{\mathbf{f}}(\nabla\times\nabla\times\hat{\mathbf{u}}_{\mathbf{f}}),$$

$$\langle \mathcal{J}'(\mathbf{f}),\mathbf{f}^*\rangle = \int_0^T\int_\Omega [\delta\mathbf{f} + \boldsymbol{\xi}_{\mathbf{f}}(\nabla\times\nabla\times\hat{\mathbf{u}}_{\mathbf{f}})]\,\mathbf{f}^*\,d\mathbf{x}dt, \quad \forall\,\mathbf{f}^*.$$

Theorem 2.2. *Let $(\hat{\mathbf{u}},\hat{\mathbf{f}})$ be an optimal solution for problem $(\mathcal{P})$. Then the following equality holds*

$$\delta\hat{\mathbf{f}} + \boldsymbol{\xi}_{\hat{\mathbf{f}}}(\nabla\times\nabla\times\hat{\mathbf{u}}) = 0 \tag{2.5}$$

where $\boldsymbol{\xi}$ is the adjoint state, solution of the adjoint linearized problem.

Thus the optimality system for problem $(\mathcal{P})$ consists of

- equation (2.1) with $\mathbf{f}=\hat{\mathbf{f}}$ and $\mathbf{u}=\hat{\mathbf{u}}$,
- equation (2.4) with $\boldsymbol{\xi}=\hat{\boldsymbol{\xi}}$, $h=\nabla\times\nabla\times\hat{\mathbf{u}}$, and $\mathbf{f}=\hat{\mathbf{f}}$

- equation (2.5).

The above set of equations is not easy to solve. One may try to compute $\hat{\mathbf{u}}$, $\hat{\mathbf{f}}$, $\hat{\boldsymbol{\xi}}$ by using optimization algorithms such as the gradient or conjugate gradient algorithm. Unfortunately, for realistic flows, these algorithms necessitate a computing power beyond that presently available. However, many researchers recently have been tried to find instead so called piecewise optimal solution or pseudo optimal solution. We will treat this problem in §4. In this case one need to solve a steady-state optimal control problem.

2.2. **The steady-state problem.** Now we consider the following steady-state problem: find the state variable and control $(\mathbf{u}, p, \mathbf{f})$ which minimized the cost functional

$$\mathcal{J}^s(\mathbf{u}, p, \mathbf{f}) = \frac{1}{2}\|\nabla \times \mathbf{u}\|^2 + \frac{\delta}{2}\|\mathbf{f}\|^2 \tag{2.6}$$

subject to the steady-state Navier-Stokes equation

$$\begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in} \quad \Omega, \\ \nabla\cdot\mathbf{u} = 0 \quad \text{in} \quad \Omega, \\ \mathbf{u} = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{2.7}$$

With $\mathcal{J}^s(\cdot,\cdot)$ given by (2.2), the admissibility set $\mathcal{U}_{ad}$ is given by

$$\mathcal{U}_{ad} = \big\{(\mathbf{u}, p, \mathbf{f}) \in \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times L^2(\Omega) \ : \ \mathcal{J}^s(\mathbf{u}, p, \mathbf{f}) < \infty \\ \text{and } (\mathbf{u}, p, \mathbf{f}) \text{ satisfies (2.1)}\big\}.$$

Then, $(\mathbf{u}, p, f) \in \mathcal{U}_{ad}$ is called an optimal solution if there exists $\epsilon > 0$ such that

$$\mathcal{J}^s(\mathbf{u}, p, \mathbf{f}) \le \mathcal{J}^s(\mathbf{v}, q, \mathbf{g}), \quad \forall\ (\mathbf{v}, q, \mathbf{g}) \in \mathcal{U}_{ad}$$

satisfying

$$\|\mathbf{u} - \mathbf{v}\|_{\mathbf{H}^1(\Omega)} + \|p - q\|_{L^2(\Omega)} + \|\mathbf{f} - \mathbf{g}\|_{L^2(\Omega)} \le \epsilon \tag{2.8}$$

The optimal control problem can now be formulated as a constrained minimization in a Hilbert space:

$\mathcal{P}_2$**:** $\min_{(\mathbf{v},q,\mathbf{g})\in\mathcal{U}_{ad}} \mathcal{J}^s(\mathbf{v}, q, \mathbf{g})$.

Theorem 2.3. *There exists an optimal control solution* $(\hat{\mathbf{u}}, \hat{p}, \hat{\mathbf{f}}) \in \mathcal{U}_{ad}$.

We wish to use the method of Lagrange multipliers to turn our constrained optimization problem into an unconstrained one. First, one should show that suitable Lagrange multipliers exist.

Theorem 2.4. *Let* $(\hat{\mathbf{u}}, \hat{p}, \hat{\mathbf{f}}) \in \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times L^2(\Omega)$ *denote an optimal solution of the optimization problem ($\mathcal{P}_2$). Then there exists a nonzero Lagrange multiplier* $(\boldsymbol{\xi}, \phi, \boldsymbol{\tau}) \in \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times \mathbf{L}^2(\Omega)$ *satisfying the adjoint equation*

$$\begin{cases} -\nu\Delta\boldsymbol{\xi} + (\nabla\hat{\mathbf{u}})^t\cdot\boldsymbol{\xi} - (\hat{\mathbf{u}}\cdot\nabla)\boldsymbol{\xi} + \nabla\phi = \nabla\times\nabla\times\hat{\mathbf{u}} \quad \textit{in} \quad \Omega, \\ \nabla\cdot\boldsymbol{\xi} = 0 \quad \textit{in} \quad \Omega, \\ \boldsymbol{\xi} = 0 \quad \textit{on} \quad \partial\Omega, \end{cases} \tag{2.9}$$

One also can prove that using $\boldsymbol{\xi} = \boldsymbol{\xi}(\nabla \times \nabla \times \hat{\mathbf{u}}_{\mathbf{f}})$,

$$\mathcal{J}'(\mathbf{f}) = \delta \mathbf{f} + \boldsymbol{\xi}_{\mathbf{f}}(\nabla \times \nabla \times \hat{\mathbf{u}}_{\mathbf{f}}). \tag{2.10}$$

Thus, the optimality condition is

$$\delta \mathbf{f} + \boldsymbol{\xi}_{\mathbf{f}}(\nabla \times \nabla \times \hat{\mathbf{u}}_{\mathbf{f}}) = 0. \tag{2.11}$$

3. Iteration Methods

The optimality system of equations consists three groups of equations: the state equations for $(\mathbf{u}, \mathbf{f})$, the adjoint state equation for $(\boldsymbol{\xi}, \mathbf{u})$, and the optimality condition for $(\mathbf{f}, \boldsymbol{\xi})$. A simple minded algorithm is given as follows:

1. Choose an initial guess $\mathbf{f}_0$;
2. for each $n \geq 1$,
   solve for $\mathbf{u}_n$ from the state equation with $\mathbf{f}_{n-1}$;
   solve for $\boldsymbol{\xi}_n$ from the adjoint equation with $\mathbf{u}_n$;
   and solve for $\mathbf{f}_n$ from the optimality condition with $\boldsymbol{\xi}_n$.

This algorithm turns out to be, at least formally, the gradient method with unit step-length for minimizing $\mathcal{K}(\mathbf{f}) := \mathcal{J}^s(\mathbf{u}(\mathbf{f}), \mathbf{f})$.

A gradient method for minimizing a functional $\mathcal{K}(\cdot)$ defined on Hilbert space H may be described as follows:

1. choose $\mathbf{f}_0$;
2. for $n = 0, 1, 2, \ldots$, set $\mathbf{f}_n = \mathbf{f}_{n-1} - \rho_n \mathcal{R} \dfrac{d\mathcal{K}(\mathbf{f}_n)}{d\mathbf{f}}$,

where $\mathcal{R}$ is the Riesz map from H^* to H and ρ_n is a sequence of positive step lengths. This algorithm converges if the second order derivative is positive definite and $0 < \rho_* \leq \rho_n \rho^* < \dfrac{2m}{M^2}$ where $M > m > 0$ are the bounds for the second order derivative in a neighborhood of an optimal solution.

This gradient algorithm for $(\mathcal{P}_2)$ consists in defining recursively a sequence of $\mathbf{f}_n \in \mathbf{L}^2(\Omega)$. Starting the initial guesses $\mathbf{f}_0$, let

$$\mathbf{f}_{n+1} = \mathbf{f}_n - \rho_n \frac{\mathcal{J}^s(\mathbf{f}_n)}{d\mathbf{f}},$$

i.e. from the equation (2.10)

$$\mathbf{f}_{n+1} = \mathbf{f}_n - \rho_n(\delta \mathbf{f}_n + \boldsymbol{\xi}_n).$$

The number $\rho_n > 0$ must be chosen properly. Thus we have

1. choose an initial guess $\mathbf{f}_0$;
2. for each $n \geq 1$,
   solve for $(\mathbf{u}_n, p_n)$ from

$$\begin{cases} -\nu \Delta \mathbf{u}_n + (\mathbf{u}_n \cdot \nabla)\mathbf{u}_n + \nabla p_n = \mathbf{f}_{n-1} \text{ in } \Omega, \\ \nabla \cdot \mathbf{u}_n = 0 \text{ in } \Omega, \\ \mathbf{u}_n = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.1}$$

   solve for $(\boldsymbol{\xi}_n, \phi_n)$ from

$$\begin{cases} -\nu \Delta \boldsymbol{\xi}_n + (\nabla \mathbf{u}_n)^t \cdot \boldsymbol{\xi}_n - (\mathbf{u}_n \cdot \nabla)\boldsymbol{\xi}_n + \nabla \phi_n = \nabla \times \nabla \times \mathbf{u}_n \text{ in } \Omega, \\ \nabla \cdot \boldsymbol{\xi}_n = 0 \text{ in } \Omega, \\ \boldsymbol{\xi}_n = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.2}$$

and upgrade $\mathbf{f}_n$ from

$$\mathbf{f}_n = (1 - \delta\rho_n)\mathbf{f}_{n-1} - \rho_n \boldsymbol{\xi}_n. \tag{3.3}$$

4. Piecewise optimal control

In the time-dependent case we will find suboptimal optimal solution. We look for a candidate velocity-pressure couple $(\mathbf{u}, p)$ by controlling external force $\mathbf{f}$ piecewise in time such that $\mathbf{u}$ best minimize the vorticity. We try to match the velocity fields at a sequence of time intervals. Precisely, we study the following piecewise (in time) optimal control problem.

1. First, choose a sufficiently small $\delta > 0$, choose a sequence $\{t_n\}_{n=0}^{N}$ defined by $t_n = n\delta$ and define $\mathbf{u}^{(0)} = \mathbf{u}_0$.
2. Then, inductively, for each n find solution $(\mathbf{u}^{(n)}, p^{(n)}, \mathbf{f}^{(n)})$ on the interval (t_{n-1}, t_n) which minimizes the instantaneous cost functional

$$\mathcal{J}^s_{t_n}\big(\mathbf{u}(t_n,\cdot), p(t_n,\cdot), \mathbf{f}(t_n,\cdot)\big) = \frac{1}{2}\left\|\nabla \times \mathbf{u}(t_n,\cdot)\right\|^2 + \frac{\delta}{2}\left\|\mathbf{f}(t_n,\cdot)\right\|^2 \tag{4.1}$$

subject to the

$$\begin{cases} \dfrac{\partial \mathbf{u}(t_n,\cdot)}{dt} - \nu\Delta\mathbf{u}(t_n,\cdot) + (\mathbf{u}(t_n,\cdot)\cdot\nabla)\mathbf{u}(t_n,\cdot) + \nabla p(t_n,\cdot) = \mathbf{f}(t_n,\cdot) \text{ in } \Omega, \\ \nabla\cdot\mathbf{u}(t_n,\cdot) = 0 \text{ in } \Omega, \\ \mathbf{u}(t_n,\cdot) = 0 \text{ on } \partial\Omega, \\ \mathbf{u}(t_{n-1},\cdot) = \mathbf{u}^{(n-1)}(t_{n-1}) \text{ in } \Omega. \end{cases} \tag{4.2}$$

We define a global (in time) solution $(\mathbf{u}, p, \mathbf{f})$ by patching together all the local optimal control solutions $(\mathbf{u}^{(n)}, p^{(n)}, \mathbf{f}^{(n)})$. One may have an algorithm for solving this problem with finite element discretization in space and first-order finite difference discretization in time.

It should be noted that the piecewise-in-time optimal control problem is different from the global optimal control problem which is defined in §2.

The piecewise-in-time optimal control problem can be thought of as the minimization of $\nabla \times \mathbf{u}$ and $\mathbf{f}$ in some $L^\infty-$norm, whereas the global optimal control problem is the minimization of these quantities in the $L^2(0,T)-$norm. We would like to point out here that the control obtained from the piecewise-in-time optimal control problem is suboptimal and there is no guarantee that it will be the same as that obtained from the global-in-time optimal control problem. However, the piecewise optimal control approach does a very good job in tracking velocity problems. The piecewise-in-time optimal control problem can be solved by marching in time and thus requires essentially the same computer storage as the steady state optimal control problem. In contrast, the numerical solution of the global optimal control problem involves a time-dependent optimality system has to be solved either with full time-space storage or by some iterative scheme that uncouples the initial and terminal conditions. In any case, it seems that in the context of flow matching and some other situations the numerical solution of the piecewise optimal control problem is more straightforward and efficient than that of the global optimal control problem.

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