Trends in Mathematics
Information Center for Mathematical Sciences
Volume 1, Number 1, September 1998, Pages 208–214

A SURVEY FOR THE COMPRESSIBLE VISCOUS NAVIER-STOKES EQUATIONS

JAE RYONG KWEON

ABSTRACT. In this note we summarize a regularity result for the whole Navier-Stokes equations in terms of historic view, especially given by da Veiga([11],[13]). A L^p-regularity estimate for the solution is given, and the proof is based on an existence result for a linearized problem and the incompressible Stokes problem, followed by the Schauder fixed point theory.

1. STATIONARY NAVIER-STOKES FLOWS

We consider the whole Navier-Stokes equations

$$\begin{cases} -\mu\Delta\mathbf{u} - \nu\nabla\mathrm{div}\,\mathbf{u} + \nabla p(\rho,\zeta) = \rho[\mathbf{f} - (\mathbf{u}\cdot\nabla)\mathbf{u}] \quad \text{in } \Omega, \\ \mathrm{div}(\rho\mathbf{u}) = g \quad \text{in } \Omega, \\ -\chi\Delta\zeta + c_\nu\rho\mathbf{u}\cdot\nabla\zeta + \zeta p'_\zeta(\rho,\zeta)\mathrm{div}\,\mathbf{u} = \rho\, h + \psi(\mathbf{u},\mathbf{u}), \quad \text{in } \Omega, \\ \mathbf{u} = 0 \quad \text{on } , , \ \zeta = \zeta_0 \quad \text{on } , , \end{cases} \tag{1}$$

Here

- $\mathbf{f}$ and h are the assigned external force field and heat sources per unit mass, and g is not zero in mathematical point of view and

$$\psi(\mathbf{u},\mathbf{u}) = \chi_0 \sum_{i,j=1}^{n} (\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})^2 + \chi_1(\mathrm{div}\,\mathbf{u})^2.$$

- $\mathbf{u}$ is a velocity vector.
- $p = p(\rho,\zeta)$ is the pressure.
- $\rho(\mathbf{x})$=the density of the fluid.
- $\zeta(\mathbf{x})$=the absolute temperature.
- $\mu > 0$, $\mu + \nu > 0$, $\chi > 0$, C_ν, χ_0, χ_1 are constants

We assume that the total mass of fluid inside Ω is fixed,i.e,

$$\frac{1}{|\Omega|}\int_\Omega \rho(\mathbf{x})\,\mathrm{d}\mathbf{x} = m, \tag{2}$$

where m is a given positive constant.

1991 *Mathematics Subject Classification.* Primary 35M10,35Q30,76N10.

Key words and phrases. compressible viscous Navier-Stokes flows, Conservation equations, hyperbolic flow, Incompressible Stokes flows.

We write and change the variables (see the papers [11, 13] for details)

$$\begin{aligned} \rho &= m+\sigma, \ \int_\Omega \sigma(\mathbf{x})\mathrm{d}\mathbf{x} = 0, \\ \zeta &= \zeta_0 + \alpha, \\ p'_\rho(\rho,\zeta) &= k + \omega_1(\sigma), \ k = p'(m,\zeta_0) > 0, \\ p'_\zeta(\rho,\zeta) &= \omega_2(\sigma,\alpha). \end{aligned}$$

Equivalent system

$$(3) \qquad \begin{cases} -\mu\Delta\mathbf{u} - \nu\nabla\mathrm{div}\,\mathbf{u} + k\nabla\sigma = F(\mathbf{f},\mathbf{u},\sigma,\alpha), & \text{in } \Omega \\ m\mathrm{div}\,\mathbf{u} + \mathbf{u}\cdot\nabla\sigma + \sigma\mathrm{div}\,\mathbf{u} = g \ \text{ in } \Omega, \\ -\chi\Delta\alpha = H(h,u,\sigma,\alpha), \ \text{ in } \Omega, \\ \mathbf{u} = 0 \ \text{ on } , , \ \alpha = 0 \ \text{ on } , , \end{cases}$$

where $F(\mathbf{f},\mathbf{u},\sigma,\alpha) = (m+\sigma)[\mathbf{f}-(\mathbf{u}\cdot\nabla)\mathbf{u}] - \omega_1(\sigma,\alpha)\nabla\sigma - \omega_2(\sigma,\alpha)\nabla\alpha$ and $H(h,\mathbf{u},\sigma,\alpha) =$ .... (see the paper[11] for details)

Sobolev Spaces

- $H^{k,p}$=the Sobolev space endowed with the usual norm $\|\cdot\|_{k,p}$
- $H_0^{k,p} = \{\mathbf{v} \in H^{k,p} : \mathbf{v} = 0 \text{ on } , \}$
- $\bar{H}^{k,p} = \{q \in H^{k,p} : \bar{q} = 0\}$
- $\bar{H}_0^{k,p} = H_0^{k,p} \cap \bar{H}^{k,p}$
- $H_{0,d}^{k,p} = \{\mathbf{v} \in H_0^{k,p} : \mathrm{div}\,\mathbf{v} = 0 \text{ on } , \}$
- If $p = 2$, write $H^k = H^{k,2}$ and similar for others.

: We cite the following regularity result by da Veiga([11]):

Theorem 1. *Let $p \in (1,\infty)$ and $j \geq -1$ satisfy $j+2 > n/p$. Assume that , $\in C^{3+j}$ and $p(\rho,\zeta) \in C^{3+j}$. There exist positive constants C_1 and C_2 such that if $\mathbf{f} \in H^{j+1,p}$ and $g \in \bar{H}_0^{j+2,p}$, and $h \in H^{j+1,p}$ satisfy*

$$\|\mathbf{f}\|_{j+1,p} + \|g\|_{j+2,p} + \|h\|_{j+1,p} \leq C_1,$$

then there exist a unique solution $(\mathbf{u},\sigma,\zeta) \in H_0^{j+3,p} \times H^{j+2,p} \times H^{j+3,p}$ of problem (3), in the ball

$$(4) \qquad \|\mathbf{u}\|_{j+3,p} + \|\rho - m\|_{j+2,p} + \|\zeta - \zeta_0\|_{j+3,p} \leq C_2.$$

2. Sketch of the Proof

Step 1(Linearization)

$$(5) \qquad \begin{cases} -\mu\Delta\mathbf{u} - \nu\nabla\mathrm{div}\,\mathbf{u} + k\nabla\sigma = F, & \text{in } \Omega \\ m\,\mathrm{div}\,\mathbf{u} + \mathbf{v}\cdot\nabla\sigma + \sigma\mathrm{div}\,\mathbf{v} = g \ \text{ in } \Omega, \\ \mathbf{u} = 0 \ \text{ on } , , \end{cases}$$

where $\mathbf{v}$ is given with $\mathbf{v}|_\Gamma = 0$.

Step 2(Historical View)

- 1958 K. O. Friedrichs(Symmetric Positive Linear Differential Equations in Comm. Pure Appl. Math., Vol. XI)

Tried to handle equations which are partly elliptic, partly hyperbolic, such as the Tricomi equation

$$(y\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2})u = 0.$$

- 1960 P. D. Lax and R. S. Phillips(Local Boundary Conditions for Dissipative Symmetric Linear Differential Operators in Comm. Pure Appl. Math., Vol. XIII)

  The following first order linear differential operator was considered:

$$L = \sum_{j=1}^{m} A^j \frac{\partial}{\partial x^j} + B \tag{6}$$

  where A^j and B are $n \times n$ matrices(not necessarily symmetric) and the A^j being continous, piecewise continuously differentiable functions of x and the B being piecewise continous in x.

  The main result obtained is: L=**a formally dissipative symmetric operator (**$B + B^* - \sum_{j=1}^{m} A^j_{x^j} \le 0$**),** Ω**= a domain whose boundary is of class** C^2**, and** $N(x)$**=smoothly varying boundary spaces which are maximal nonpositive(**$uA_n u \le 0,\ u \in N(x),\ x \in\ ,$ **). Then** $\forall f \in L^2(\Omega)$**, the equation** $u - Lu = f$ **has a unique strong solution satisfying in the strong sense the given boundary condition with** $\|u\| \le \|(I-L)u\|$.
- 1981 S. Klainerman and A. Majda(Singular limits of Quasilinear Hyperbolic Systems with Large Parameters and the Incomprerssible Limit of Compressible Fluids in Comm. Pure Appl. Math., Vol. XXXIV)

  Gave an entirely self-contained classical proof of the convergence of solutions for the compressible fluid equations to their incompressible limits as the Mach number becomes small.
- 1983 M. Padula(Existence and Uniqueness for Viscous Steady Compressible Motions in Arch. Rational. Mech. Anal.)

  The aim of this paper is to show the existence and uniqueness of steady flow for a viscous, isothermal, compressible fluid.
- 1984 A. Valli(Periodic and Stationary Solutions for Compressible Navier-Stokes Equations via a Stability method, Ann. Scuola Normale Sup. Pisa)
- 1986 A. Valli and W. M. Zajaczkowski(Navier-Stokes Equations for Compressible Fluids: Global Existence and Qualitative Properties of the Solutions in the General Case in Commun. Math. Phys., 103)

  Considered the equations which describe the motion of a viscous compressible fluid, taking into consideration the case of inflow and or outflow through the boundary, and showed that there exist a periodic solution and a stationary solution.
- 1987 H. Beirao da Veiga(An L^p-Theory for the n-Dimensional, Stationary, Compressible Navier-Stokes Equations, and the Incompressible Limit for Compressible Fluids. The Equilibrium Solutions in Commun. Math. Phys., 109)

– **In this paper, the stationary motion of a compressible heat conducting, viscous fluid in a bounded domain Ω of R^n, $n \geq 2$ was considered.**

1. The main ideas used are following:
   (a) Applying the divergence operator to both sides $(5)_a$ one has that

$$(\mu+\nu)\Delta \mathrm{div}\,\mathbf{u} = k\Delta\sigma - \nabla\cdot\mathbf{F}$$

and applying the Laplacian operator to both sides of $(5)_b$ one gets

$$\frac{mk}{\mu+\nu}\Delta\sigma + \mathbf{v}\cdot\nabla\Delta\sigma = G(\mathbf{F}, g, \sigma),$$

where

$$G(\mathbf{F}, g, \sigma) = \Delta g + \frac{mk}{\mu+\nu}\nabla\cdot\mathbf{F} - [2\nabla\mathbf{v} : \nabla^2\tau + \Delta\mathbf{v}\cdot\nabla\tau + \Delta(\tau\nabla\cdot\mathbf{v})].$$

   (b) For given $\tau \in H^{1,p}$, consider the linear problem

$$\frac{mk}{\mu+\nu}\lambda + \mathbf{v}\cdot\nabla\lambda = G(\mathbf{F}, g, \tau). \tag{7}$$

   (c) By da Veiga([14]) there exists a positive constant c_1 such that if $\|\mathbf{v}\|_{2,p} \leq c_1\frac{mk}{\mu+\nu}$, then there exists a linear continuous map $L : H^{-1,p} \to H^{-1,p}$ such that $\lambda = LG$ is a weak solution of (7).
   (d) Estimating $\|G\|_{-1,p}$ one has

$$\begin{aligned}\frac{mk}{\mu+\nu}\|\lambda\|_{-1,p} &\leq C\|G\|_{-1,p}\\ &\leq c(\frac{mk}{\mu+\nu}|F|_p + \|g\|_{1,p} + \|\mathbf{v}\|_{2,p}\|\tau\|_{1,p}).\end{aligned}$$

   (e) Next let $\theta \in H_0^{1,p}$ be the solution of the Dirichlet problem

$$\begin{aligned}(\mu+\nu)\Delta\theta &= k\lambda - \nabla\cdot\mathbf{F} \text{ in } \Omega,\\ \theta &= 0 \text{ on },\end{aligned}$$

and one may get

$$(\mu+\nu)\|\theta\|_{1,p} \leq c|\mathbf{F}|_p + c\frac{mk}{\mu+\nu}(\|g\|_{1,p} + \|\mathbf{v}\|_{2,p}\|\tau\|_{1,p}).$$

   (f) Define $\theta_0 = \theta - \bar{\theta}$, $\bar{\theta}$ is the average value of θ.
   (g) Consider

$$\begin{cases} -\mu\Delta\mathbf{u} + k\nabla\sigma = \mathbf{F} + \nu\nabla\theta_0,\\ \mathrm{div}\,\mathbf{u} = \theta_0, \quad \text{in } \Omega,\\ \mathbf{u}|_\Gamma = 0. \end{cases} \tag{8}$$

   (h) One has

$$\begin{aligned}\mu\|\mathbf{u}\|_{2,p} + k\|\sigma\|_{1,p} &\leq c(|\mathbf{F}|_p + (\mu+\nu)\|\theta_0\|_{1,p})\\ &\leq \frac{k}{2}\|\tau\|_{1,p} + c(1 + \frac{\mu+|\nu|}{\mu+\nu})|\mathbf{F}|_p + \frac{\mu+|\nu|}{m}\|g\|_{1,p}\end{aligned}$$

(i) Finally considered the sequence of linear maps:

$$(\mathbf{F}, g, \tau) \to (\mathbf{F}, G) \to (\mathbf{F}, \lambda) \to (\mathbf{F}, \theta_0) \to (\mathbf{u}, \sigma)$$

(j) Consequently the regularity (4) was obtained.

2. For barotropic flow the stationary solution of the Navier-Stokes equations is the incompressible limit of the stationary solution of the compressible Navier-Stokes equations as the Mach number becomes small.
3. For the whole system a unique existence of the solution was shown, satisfying a corresponding regularity.

- 1987 A. Valli(On the existence of the stationary solutions to compressible Navier-Stokes equations in Ann. Inst. Henri Poincare, Vol. 4)
  - Under the assumtion that the external force field is small, the existence of a stationary solution for the compressible Navier-Stokes fluids was shown.
  - The solution satisfies the following a priori estimate:

(9)
$$\|\mathbf{u}\|_{k+1} + \|p\|_k \le C(\|\mathbf{f}\|_{k-1} + \|g\|_k), \ (k = 0, 1, 2)$$

- 1987 H. Beiro da Veiga(Stationary Motions and the Incompressible Limit for Compressible Viscous Fluids in Houston J. of Math., Vol. 13)

  In this paper, as da Veiga did in L^p space, a regularity result was obtained in the L^2 space (see (9).)

3. Other related papers

- 1987 H. Beiro da Veiga(Existence results in Sobolev spaces for a stationary transport equation in Ricerche di Matematica Suppl., Vol. XXXVI)

  Considered the following equation mainly:

$$\lambda u + (\beta \cdot \nabla)u + au = f.$$

- 1988 H. Beiro da Veiga(Kato's Perturbation Theory and Well-Posedness for the Euler Equations in Bounded Domains in Arch. Rat. Mech. Anal.)

$$\lambda u + (\beta \cdot \nabla)u + au = f.$$

4. Compressible Navier-Stokes with Inflow Boundary Condition

- 1988 R. B. Kellogg(Discontinuous Solution of the Linearized Steady State Compressible Viscous Navier-Stokes Equations in SIAM J. Math. Anal., Vol. 19)

  Considered the following compressible Navier-Stokes equations with inflow boundary condition for pressure:

(10)
$$\begin{cases} -\mu\Delta\mathbf{u} - \nu\nabla\mathrm{div}\,\mathbf{u} + \nabla p = \mathbf{f}, \\ \mathrm{div}\,\mathbf{u} + p_x = 0, \\ \mathbf{u} = 0, \quad \text{on} \ , \ , \\ p = 0, \quad \text{on} \ ,_{in}, \end{cases}$$

where Ω is the rectangle defined by $0 < x < 1, - < y < 1$.

Jump conditions across a curve of discontinuity of a solution of (10) are derived and discontinuous slution is constructed.

- 1991 S. E. Chen and R. B. Kellogg(An Interior Discontinuity of a Nonlinear Elliptic-Hyperbolic System in SIAM J. Math. Anal., Vol. 22)

  On a strip $(0, a) \times (-\infty, \infty)$, for pressure $p(0, y) = p_0(y)$ is imposed, where $p_0(y)$ has a jump at $y = 0$. Jump conditions for system show that $\mathbf{u} = (u, v)$ is continuous, but their derivatives and pressure have a curve of discontinuity, which is shown by the Schauder fixed-point theorem, with sufficiently small width of the strip.
- 1993 B. Liu and R. B. Kellogg(Discontinous Solution of Linearized Steady State Viscous Compressible Flows in J. Math. Anal. Appl., Vol. 180)

  On an infinite strip $(0, 1) \times (-\infty, \infty)$ a discontinuous solution for a linearized system is discussed.
- 1997 J. R. Kweon and R. B. Kellogg(Compressible Navier-Stokes Equations in a Bounded Domain with Inflow Bundary Condition in SIAM J. Math. Anal., Vol. 28)

  Except for energy equation, momentum and conservation equations are considered, imposing on pressure an inflow boundary condition. A weak singularity occurs at the junction of incoming and outgoing flows. A regularity result is obtained (see the paper[19] for details):

$$\|\mathbf{u} - \mathbf{u}_0\|_{2,q} + \|p - p_0\|_{1,q} \leq C(\|\mathbf{f}\|_{0,q} + \|g\|_{1,q}), \ (2 < q < 3) \tag{11}$$

- 1998 J. R. Kweon and R. B. Kellogg(Smooth Solution of the Compressible Navier-Stokes Equations in an Unbounded Domain with Inflow Boundary Condition in J. Math. Anal. Appl., Vol. 220)

  On an infinite strip the whole Navier-Stokes equations, except for energy one, are studied, and some regularity results are obtained.

References

[1] R. A. Adams, *Sobolev Spaces, Academic press, New York, 1975.*

[2] K. O. Friedrichs, Symmetric Positive Linear Differential Equations in Comm. Pure Appl. Math., 1958(6).

[3] P. D. Lax and R. S. Phillips, Local Boundary Conditions for Dissipative Symmetric Linear Differential Operators in Comm. Pure Appl. Math., 1960(8).

[4] A. Matsumura, T. Nishida, The initial value problem for the equations of motion of compressible viscous and heat-conductive fulids, Proc. Jpn. Acad. Ser. A55, 1979, 337-342.

[5] A. Matsumura, T. Nishida, The initial value problem for the equations of motion of viscous and heat-conductive gases, J. Math. Kyoto Univ. (JMKYAZ), 1980(20-1).

[6] A. Matsumura, T. Nishida, Initial Value problems for the equations of motion of compressible viscous and heat-conductive fluids, Commun. Math. Phys., 1983(89).

[7] S. Klainerman and A. Majda, Singular limits of Quasilinear Hyperbolic Systems with Large Parameters and the Incomprerssible Limit of Compressible Fluids in Comm. Pure Appl. Math., 1981(34).

[8] M. Padula, Existence and Uniqueness for Viscous Steady Compressible Motions in Arch. Rational. Mech. Anal., 1983.

[9] A. Valli, Periodic and Stationary Solutions for Compressible Navier-Stokes Equations via a Stability method, Ann. Scuola Normale Sup. Pisa, 1983(10).

[10] A. Valli and W. M. Zajaczkowski, Navier-Stokes Equations for Compressible Fluids: Global Existence and Qualitative Properties of the Solutions in the General Case in Commun. Math. Phys.,1986(103).

[11] H. Beirao da Veiga, An L^p-Theory for the n-Dimensional, Stationary, Compressible Navier-Stokes Equations, and the Incompressible Limit for Compressible Fluids. The Equilibrium Solutions in Commun. Math. Phys., 1987(109).

[12] A. Valli, On the existence of the stationary solutions to compressible Navier-Stokes equations in Ann. Inst. Henri Poincare, 1987(4).

[13] H. Beiro da Veiga, Stationary Motions and the Incompressible Limit for Compressible Viscous Fluids in Houston J. of Math., 1987(13).

[14] H. Beiro da Veiga, Existence results in Sobolev spaces for a stationary transport equation in Ricerche di Matematica Suppl., 1987(36).

[15] H. Beiro da Veiga, Kato's Perturbation Theory and Well-Posedness for the Euler Equations in Bounded Domains in Arch. Rat. Mech. Anal., 1988.

[16] R. B. Kellogg, Discontinuous Solution of the Linearized Steady State Compressible Viscous Navier-Stokes Equations in SIAM J. Math. Anal., 1988(19).

[17] S. E. Chen and R. B. Kellogg, An Interior Discontinuity of a NonlinearElliptic-Hyperbolic System in SIAM J. Math. Anal., 1991(22).

[18] B. Liu and R. B. Kellogg, Discontinous Solution of Linearized Steady State Viscous Compressible Flows in J. Math. Anal. Appl., 1993(180).

[19] J. R. Kweon and R. B. Kellogg(Compressible Navier-Stokes Equations ina Bounded Domain with Inflow Bundary Condition in SIAM J. Math. Anal., 1997(28).

[20] J. R. Kweon and R. B. Kellogg, Smooth Solution of the Compressible Navier-Stokes Equations in an Unbounded Domain with Inflow Boundary Condition in J. Math. Anal. Appl., 1998(220).

Department of Mathemtics, College of Natural Science Silla University, Pusan
E-mail address: `kweon@lotus.silla.ac.kr`