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THEORY OF NON-NEWTONIAN FLOW

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Abstract. We consider the mathematical theory of non-Newtonian fluids. We review that there exist Young measure valued solutions to non-Newtonian flows, and that the measure valued solutions are weak solutions. Galerkin approximation, $W^{1,r+2}$ and $W^{1,2}$ compactness theorems, and Korn type inequalities are main ingredients for the proof of the existence of weak solutions.

1. Introduction

Fluid dynamics has attracted the attentions of many mathematicians and engineers. The Navier-Stokes equations are generally accepted as right governing equations for the incompressible motion of viscous fluids. If the relation between the stress and the rate of strain of a fluid is linear, then the fluid is called Newtonian. That is, Newtonian fluids satisfy the linear relation

$$\tau = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

where τ is the stress and $\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$ is the rate of strain. The coefficient of proportionality μ is called the viscosity, and it is a characteristic material quantity for the fluid concerned, which in general depends on temperature and pressure. Air, other gases, water, motor oil, alcohols, simple hydrocarbon compounds and others tend to be Newtonian fluids. The governing equations of motions of them will be the Navier-Stokes equations. If the relation is not linear, the fluid is called non-Newtonian. Examples of non-Newtonian fluids are molten plastics, polymer solutions, dyes, varnishes, suspensions, adhesives, paints, greases, paper pulp, biological fluids like blood, salad dressings and butter. Indeed, non-Newtonian materials are often classified by the term *visco-elastic*, indicating that they display both the properties of viscous fluids and elastic solids. The simplest model of the stress-strain relation for such fluids is given by the power laws, which states that

$$\tau = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^q,$$

for $q > 0$, see Böhme [5].

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Ladyzhenskaya [6] proposed a new model to study some kinds of non-Newtonian fluids which is of interest to us. Ladyzhenskaya model is the following:

$$
\begin{cases}
\rho \dfrac{\partial u_i}{\partial t} + \rho u_j \dfrac{\partial u_i}{\partial x_j} = -\dfrac{\partial p}{\partial x_i} + \dfrac{\partial ,_{ij}}{\partial x_j} + \rho f_i, \\
\dfrac{\partial u_j}{\partial x_j} = 0, \\
,_{ij} \overset{\text{def}}{=} \big(\mu_0 + \mu_1 |E(\nabla u)|^r\big) E_{ij}(\nabla u), \\
E_{ij}(\nabla u) \overset{\text{def}}{=} \dfrac{1}{2}\big(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\big),
\end{cases}
\tag{1.1}
$$

in $Q = \Omega \times (0, \infty)$, the initial condition $u(x,0) = u_0$ for $x \in \Omega$ and with Dirichlet boundary condition, where $E = (E_{ij})$. Here, $\Omega \subset \mathbb{R}^n$ is bounded, and $r > -1$, and $\mu_0, \mu_1 > 0$. These models are called

$$
\begin{cases}
\text{Newtonian} & \text{for } \mu_0 > 0,\ \mu_1 = 0, \\
\text{Rabinowitsch} & \text{for } \mu_0, \mu_1 > 0, \text{ and } r = 2, \\
\text{Ellis} & \text{for } \mu_0, \mu_1 > 0, \text{ and } r > 0, \\
\text{Ostwald-de Waele} & \text{for } \mu_0 = 0,\ \mu_1 > 0, \text{ and } r > -1, \\
\text{Bingham} & \text{for } \mu_0, \mu_1 > 0, \text{ and } r = -1.
\end{cases}
$$

For $\mu_0 = 0$, if $r < 0$ then it is a pseudo-plastic fluid, and if $r > 0$ then it is a dilant fluid. see Böhme [5]. The values of the parameters μ_1, r of some of the pseudo-plastic Ostwald-de Waele models are the following; for example, for paper pulp $\mu_1 = 0.418$, $r = -0.425$, and for carboxymethyl cellulose in water $\mu_1 = 0.194$, $r = -0.434$.

2. History of Non-Newtonian Flow

Many mathematicians have tried to answer the following questions.

1. For which r do measure valued solutions exist?
2. For which r are measure valued solutions Dirac ones? In other words, For which r do weak solutions exist?
3. For which r, if any, do weak solutions have some qualitative properties such as higher regularity, uniqueness?

Ladyzhenskaya [6], and Lions [7] for $\mu_0, \mu_1 > 0$ obtained the existence of the weak solutions for r such that $r \geq 0$ for $n = 2$, and $r \geq \frac{1}{5}$ for $n = 3$, and their uniqueness for r such that $r \geq 0$ for $n = 2$, and $r \geq \frac{1}{2}$ for $n = 3$. Lions [7] considered a little different setting on the nonlinear viscosity term, which were p-Laplace operator. Bellout, Bloom and Nečas [4], and Málek, Nečas and Novotný [8] have shown that for $\mu_0, \mu_1 > 0$, there are Young measure valued solutions if $r \in (-1, -\frac{1}{2}]$ for $n = 2$, and $r \in (-\frac{4}{5}, -\frac{1}{5}]$ for $n = 3$. They also obtained, for periodic problems, the existence of weak solutions if $r > -\frac{1}{2}$ for $n = 2$, and $r > -\frac{1}{5}$ for $n = 3$, and that the weak solutions are unique regular solutions if $r \geq 0$ for $n = 2$, and $r \geq \frac{1}{5}$ for $n = 3$.

Bae and Choe [2] have shown the existence of Young measure valued solutions for all $r \in (-1, \infty)$ when $\mu_0, \mu_1 > 0$ and $n = 3$. Moreover, it is shown the Young measure valued solutions are weak solutions if certain convexity condition for energy holds. Málek, Nečas, Rokyta and Růžička [9] have shown that the weak solutions are regular for $r > -1$ when $\mu_0, \mu_1 > 0$ and $n = 2$, and Bae [1] have obtained the similar result for $r > -1$ when $\mu_0 = 0, \mu_1 > 0$ and $n = 2$. It has been open problem for the existence of weak solutions for

$$-1 < r < -\frac{1}{5},$$

see Málek, Nečas, Rokyta and Růžička [9]. In Bae and Choe [3], the existence of weak solutions of 3D non-Newtonian flow is shown for $-1 < r < 0$ when $\mu_0, \mu_1 > 0$.

3. Existence of Young measure valued solutions

In this section we review the existence of Young measure valued solutions of (1.1) for $r > -1$ given in Bae and Choe [2]. The Galerkin method is used to show the existence. The energy estimate follows in a standard way. But the difficulty lies on showing compactness in L^2-space. With the compactness we can prove the strong convergence of approximate convection terms. Consider (1.1) on $Q_T \stackrel{\text{def}}{=} \Omega \times [0, T]$, where $T > 0$ is fixed.

For the analysis of non-Newtoninan flow, the following Korn's inequalities for $1 < s < \infty$ are useful;

$$\int_\Omega |E_{ij}(\nabla v) E_{ij}(\nabla v)|^{s/2}\, dx \geq C \|\nabla v\|_s^s,$$

$$\int_\Omega \left| \frac{\partial E_{ij}(\nabla u)}{\partial x_k} \frac{\partial E_{ij}(\nabla u)}{\partial x_k} \right|^{s/2} dx \geq C \|u\|_{2,s}^s.$$

Let $\mathcal{V} \stackrel{\text{def}}{=} \{v \in C_0^\infty(\Omega)^3 : \text{ div } v = 0\}$, and define by $\mathbf{H}, \mathbf{V}, \mathbf{V}_q$, the closures of $\mathcal{V}$ in $L^2(\Omega)^3$, $W^{1,2}(\Omega)^3$, and in $W^{1,q}(\Omega)^3$, respectively. Let $\mathbb{V}$ be the dual space of $\mathbf{V}$. Since $\mathcal{V}$ is dense in $\mathbf{V}$ and in $\mathbf{V}_{r+2}$, and $\mathbf{V}$ and $\mathbf{V}_{r+2}$ are separable, there exist an orthonormal subset $\{w_i : i = 1, 2, \cdots\} \subset \mathcal{V}$ which spans $\mathbf{V} \cap \mathbf{V}_{r+2}$. For each m we define an approximate solution u^m of (1.1):

$$u^m \stackrel{\text{def}}{=} \sum_{\ell=1}^m g_\ell^m(t) w_\ell(x)$$

and

$$\langle u_t^m, w_k \rangle + \int E_{ij}(\nabla u) E_{ij}(\nabla v)\, dx + \int |E(\nabla u)|^r E_{ij}(\nabla u) E_{ij}(\nabla v)\, dx + \int_\Omega (u^m \cdot \nabla) u^m \cdot w_k\, dx = \langle f, w_k \rangle, \tag{3.1}$$

where $u^m(0) = u_0^m$ is the orthogonal projection in $\mathbf{H}$ of u_0 onto the space spanned by $\{w_1, \cdots, w_m\}$. By the theory of nonlinear ODE's, the existence and uniqueness of solution of (3.1) is obtained easily. In other words, for each $m \geq 1$, (3.1) has a maximal solution on some interval $[0, t_m)$. If $t_m < T$, then $\|u^m(t)\| \to \infty$ as $t \to t_m$. The following lemma shows that this does not happen, therefore $t_m = T$.

Lemma 3.1. *Assume that $\int \|f\|_{\mathbb{V}}^2\, dt$ is bounded. Then we have*

$$\sup_{0<t<T} \|u^m(t)\|^2 + \mu_0 \int_0^T \|\nabla u^m\|^2 ds + \mu_1 \int_0^T \|\nabla u^m\|_{r+2}^{r+2} ds \leq C \|u^m(0)\|^2 + C \int_0^T \|f\|_{\mathbb{V}}^2 ds.$$

From the above theorem, we have

$$u^m \in L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V}) \cap L^{r+2}(0,T;\mathbf{V}_{r+2}).$$

Thus, we have a subsequence, still denoted by u^m, which converges to u in $L^2(0,T;\mathbf{V})$ weakly, in $L^{r+2}(0,T;\mathbf{V}_{r+2})$ weakly, and in $L^\infty(0,T;\mathbf{H})$ weak-star, as $m \to \infty$. For such a subsequence we need to take the limit to (3.1) in order to show the limit u is a Young measure valued solution. Owing to the above statement, we can take the limit to the infinity for the first term of (3.1) and for the linear viscosity term. For the nonlinear third term of (3.1) we need the strong convergence of u^m in $L^2(0,T;\mathbf{H})$. For the nonlinear viscosity term we need to find the corresponding Young measure to the weak limit of ∇u^m.

For the strong convergence of the subsequence u^m in $L^2(0,T;\mathbf{H})$, we need a compactness lemma. We state the compact embedding lemma in Temam [10] (see Theorem 2.2 of Ch. III in [10]).

Lemma 3.2. *Assume that X_0, X, X_1 are Hilbert spaces with*

$$X_0 \subset X \subset X_1,$$

the injection being continuous and the injection of X_0 into X is compact. Define the space

$$\mathcal{H}^\gamma(\mathbb{R};X_0,X_1) \stackrel{\text{def}}{=} \{v \in L^2(\mathbb{R};X_0), \partial_t^\gamma v \in L^2(\mathbb{R},X_1)\},$$

which is a Hilbert space for the norm

$$\|v\|_{\mathcal{H}^\gamma(\mathbb{R};X_0,X_1)} \stackrel{\text{def}}{=} \{\|v\|^2_{L^2(\mathbb{R};X_0)} + \|\,|\tau|^\gamma \hat{v}(\tau)\|^2_{L^2(\mathbb{R};X_1)}\}^{1/2},$$

where $\hat{v}$ is the Fourier transform of v with respect to t

$$\hat{v}(\tau) = \int_{\mathbb{R}} e^{-2\pi i t\tau} v(t)\,dt.$$

We denote $i = \sqrt{-1}$ if there is no confusion with index i.

We also define a subspace $\mathcal{H}^\gamma_K$ of $\mathcal{H}^\gamma$ for any $K \subset \mathbb{R}$ defined as the set of functions u in $\mathbf{H}^\gamma$ with support contained in K:

$$\mathcal{H}^\gamma_K(\mathbb{R};X_0,X_1) \stackrel{\text{def}}{=} \{u \in \mathcal{H}^\gamma(\mathbb{R};X_0,X_1),\ \text{support } u \subset K\}.$$

Then, for any bounded set K and any $\gamma > 0$, the injection of $\mathcal{H}^\gamma_K(\mathbb{R};X_0,X_1)$ into $L^2(\mathbb{R};X)$ is compact.

Lemma 3.3. *Suppose $\mu_0, \mu_1 > 0$. Let $-1 < r$. If, for each m, u^m is a solution of (3.1), then $u^m \in \mathcal{H}^\gamma_{[0,T]}(\mathbb{R};\mathbf{V},\mathbf{H})$ for $0 < \gamma < 1/4$.*

By using Lemma 3.1, Lemma 3.2 and Lemma 3.3, we obtain the existence of Young measure valued solutions.

Theorem 3.4 (Existence of Young measure valued solutions)**.** *Let μ_0, $\mu_1 > 0$. Let $r > -1$. If $u_0 \in \mathbf{H}$ and $f \in L^2(0,T;\mathbb{V})$, then there is a measure valued solution $(u,\nu_{x,t})$ of (1.1) satisfying*

$$-\iint_{Q_T} u_i \partial_t \phi_i \,dx\,dt - \iint_{Q_T} u_j u_i \nabla_j \phi_i \,dx\,dt + \mu_0 \iint_{Q_T} E_{ij}(\nabla u) E_{ij}(\nabla \phi)\,dx\,dt$$
$$+ \mu_1 \iint_{Q_T} \int_{\mathbb{R}^3\times\mathbb{R}^3} |E(\lambda)|^r E(\lambda)\,d\nu_{x,t}(\lambda) \cdot E(\nabla\phi)\,dx\,dt = \iint_{Q_T} f_i \phi_i \,dx\,dt,$$

for all $\phi \in C_0^\infty(Q_T)$ *with* $\nabla \cdot \phi = 0$*, where*

$$E_{ij}(\lambda) = \frac{1}{2}\left(\lambda_{ij} + \lambda_{ji}\right).$$

4. Weak solutions for $r > -1$

In this section we show the Young measure valued solutions for the periodic problems obtained in the previous section are weak for $-1 < r < 0$.

We have obtained a Galerkin approximate solution u^m of (3.1). For the limiting process in the nonlinear viscosity term of (3.1), for some $u \in L^{r+2}(0,T;\mathbf{V}_{r+2})$ and for all $\phi \in C_0^\infty(-\infty,T;\mathcal{V})$,

$$\iint_{Q_T} |E(\nabla u^m)|^r E_{ij}(\nabla u^m)\, E_{ij}(\nabla\phi)\, dx\, dt \to \iint_{Q_T} |E(\nabla u)|^r E_{ij}(\nabla u)\, E_{ij}(\nabla\phi)\, dx\, dt \tag{4.1}$$

should be justified. If we can prove that for some $s \geq 1$, ∇u^m converges to ∇u strongly in $L^s(Q_T)^{3\times 3}$, then $\nabla u^m \to \nabla u$ almost everywhere in Q_T and also

$$|E(\nabla u^m)|^r \nabla u^m \to |E(\nabla u)|^r \nabla u, \qquad \text{a.e. in } Q_T.$$

Then, for all $M \subset Q_T$,

$$\int_M |E(\nabla u^m)|^{r+1}\, dx\, dt \leq C\Big(\int_0^T \int_\Omega |E(\nabla u^m)|^{r+2}\, dx\, dt\Big)^{\frac{r+1}{r+2}} |M|^{\frac{1}{r+2}} \leq C|M|^{\frac{1}{r+2}}.$$

By applying Vitali theorem we obtain the convergence (4.1). For details, refer to Málek, Nečas, Rokyta and Růžička [9]. Therefore, we need the strong convergence of u^m in $L^s(Q_T)^{3\times 3}$ for some $s \geq 1$. For this, we need a compactness lemma in $L^s(0,T;W^{1,r+2}(\Omega))$ for some $s > 1$. We state the compact embedding lemma (Aubin-Lions theorem) in Theorem 2.1 of Ch. III in Temam [10].

Lemma 4.1. *Assume that* Y_0, Y, Y_1 *are Banach spaces with*

$$Y_0 \subset Y \subset Y_1,$$

the injection being continuous and the injection of Y_0 *into* Y *is compact, and* Y_0 *and* Y_1 *are reflexive. Let* $T > 0$ *be finite, and let* $\alpha_0, \alpha_1 > 1$ *be finite numbers. Define the space*

$$\mathcal{Y} \stackrel{\text{def}}{=} \{v \in L^{\alpha_0}(0,T;Y_0),\ \dot{v} \stackrel{\text{def}}{=} \frac{d}{dt}v \in L^{\alpha_1}(0,T;Y_1)\},$$

which is a Banach space for the norm

$$\|v\|_{\mathcal{Y}} \stackrel{\text{def}}{=} \|v\|_{L^{\alpha_0}(0,T;Y_0)} + \|\dot{v}\|_{L^{\alpha_1}(0,T;Y_1)}.$$

Then, the injection of $\mathcal{Y}$ *into* $L^{\alpha_0}(0,T;Y)$ *is compact.*

We now prove that $\{u^m\}$ lies in $\mathcal{Y}$ for some $\alpha_0, \alpha_1 > 1$. The constants σ, α_0 and α_1 will be determined later.

Lemma 4.2. *Let α_1 any number such that $1 < \alpha_1 \le 2$. Assume that*

$$\int_0^T \|f\|_{\mathbb{V}}^{\alpha_1} dt$$

is bounded. Then we have

$$\dot{u}^m(t) \in L^{\alpha_1}(0,T; \left(W^{2,2}(\Omega)\right)^*).$$

The following lemma will be useful for the proof of the existence of weak solutions.

Lemma 4.3.

$$\int |\nabla^2 u|^{r+2} dx \ \le \ C \int_\Omega \partial_k(|E(\nabla u)|^r E_{ij}(\nabla u))\partial_k E_{ij}(\nabla u)\, dx + C \int |\nabla u|^{r+2} dx.$$

Consider the inner product of (3.1) with $\dfrac{-\partial_k^2 u^m}{(1+\|\nabla u^m\|^2)^\lambda}$, where λ is a number ≥ 2. Integrating with respect to t and by applying Young's Hölder's, and Sobolev's inequalities, we obtain

$$\begin{aligned}(4.2) \quad & \frac{1}{2(1-\lambda)}(1+\|\nabla u^m(T)\|^2)^{1-\lambda} + C \iint \frac{\partial_k(|E|^r E)\partial_k E}{(1+\|\nabla u^m\|^2)^\lambda}\, dx\, dt \\ & + C \iint \frac{|\nabla^2 u^m|^2}{(1+\|\nabla u^m\|^2)^\lambda}\, dx\, dt \le \iint \frac{|\nabla u^m|^3 + |f|^2}{(1+\|\nabla u^m\|^2)^\lambda}\, dx\, dt \\ & + \frac{1}{2(1-\lambda)}(1+\|\nabla u^m(0)\|^2)^{1-\lambda}.\end{aligned}$$

By applying Hölder's, Sobolev's, Korn's inequalities to the term containing $|\nabla u^m|^3$, and Lemma 4.3, we derive

$$(4.3) \qquad \iint \frac{|\nabla^2 u^m|^{r+2}}{(1+\|\nabla u^m\|^2)^\lambda}\, dx\, dt + \iint \frac{|\nabla^2 u^m|^2}{(1+\|\nabla u^m\|^2)^\lambda}\, dx\, dt \le C.$$

By the interpolation inequality and (4.3), we obtain that for some $s > 1$ and $0 < \sigma < 1$,

$$\int \|u^m\|_{1+\sigma, r+2}^s\, dt \le C.$$

By Sobolev embedding, we have

$$W^{1+\sigma,r+2} \subset W^{1,r+2} \subset (W^{2,2})^*,$$

of which the first inclusion is compact. Therefore, we can apply Lemma 4.1 where $Y_0 = W^{1+\sigma,r+2}$, $Y = W^{1,r+2}$, and $Y_1 = (W^{2,2})^*$, and α_0 can be chosen by s. Hence, by the Aubin-Lions Theorem, the sequence $\{u^m\}$ is compact in $L^{\alpha_0}(0,T;\mathbf{V}_{r+2})$. Thus there exists a subsequence, still denoted by u^m, converging strongly in $L^{\alpha_0}(0,T;W^{1,r+2})$. Therefore, we have our main theorem on the weak solutions.

Theorem 4.4 (Weak solutions)**.** *Let $\mu_0, \mu_1 > 0$. Let r be a number with $-1 < r$. If $u_0 \in \mathbf{H}$ and $\int_0^T \|f\|_{\mathbb{V}}^2 dt$ is bounded, then the Young measure valued solution*

$(u, \nu_{x,t})$ of (1.1) becomes a weak solution satisfying

$$-\iint_{Q_T} u_i \partial_t \phi_i \, dx\, dt + \mu_0 \iint_{Q_T} E(\nabla u) \cdot E(\nabla \phi)\, dx\, dt$$
$$- \iint_{Q_T} u_j u_i \nabla_j \phi_i \, dx\, dt + \mu_1 \iint_{Q_T} |E(\nabla u)|^r E(\nabla u) \cdot E(\nabla \phi)\, dx\, dt$$
$$= \int f_i \phi_i \, dx\, dt,$$

for all $\phi \in C_0^\infty(Q_T)$ with $\nabla \cdot \phi = 0$. and $Q_T \stackrel{\text{def}}{=} \Omega \times [0, T]$. Moreover, the Young measure $\nu_{x,t}(\lambda) = \delta(\lambda - \nabla u(x,t))$ a.e. in Q_T.

References

[1] H.-O. Bae, *Existence, regularity and decay rate of solutions of non-Newtonian Flow*, submitted

[2] H.-O. Bae and H. J. Choe, *Existence and regularity of solutions of non-Newtonian Flow*, to appear in Quart. Appl. Math.

[3] H.-O. Bae and H. J. Choe, *Existence of Weak Solutions to a Class of Non-Newtonian Flows*, RIM-GARC Preprint Series 98-43

[4] H. Bellout, F. Bloom and J. Nečas, *Young measure-valued solutions for non-Newtonian incompressible fluids*, Comm. Part. Diff. Eqs., 19(11&12), 1763-1803 (1994)

[5] G. Böhme, Non-Newtonian Fluid Mechanics, North-Holland Series in Applied Mathematics and Mechanics, 1987

[6] O. A. Ladyzhenskaya, *New equations for the description of the viscous incompressible fluids and solvability in the large of the boundary value problems for them*, Boundary Value Problems of Mathematical Physics V, Amer. Math. Soc., Providence, RI, (1970)

[7] J. L. Lions, Quelques Méthodes de Résolution des Problèms aux Limites Non Linéaires, Dunrod, 1969

[8] J. Málek, J. Nečas and A. Novotný, *Measure valued solutions and asymptotic behaviour of a multipolar model of a boundary layer*, Czechoslovak Math. J. **42**(117), (1992), 549-576

[9] J. Málek, J. Nečas, M. Rokyta and M. Růžička, Weak and Measure-valued Solutions to Evolutionary PDEs, Chapman & Hall, 1996

[10] R. Temam, Navier-Stokes Equations, Elsevier, 1977

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