Trends in Mathematics
Information Center for Mathematical Sciences
Volume 1, Number 1, September 1998, Pages 000–000

NUMERICAL COMPUTATIONS FOR INCOMPRESSIBLE NAVIER-STOKES EQUATIONS USING FINITE ELEMENT METHOD

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Abstract. This survey paper is devoted to the computational fluid dynamics(CFD) for Incompressible Navier-Stokes equations based on finite element methods. There are two major difficulties in CFD using finite element methods. One is pressure correction and another is to resolve the nonlinearity, the former of which will be discussed based on variants of Uzawa algorithm. Also the solver has to be newly designed by upwind scheme for its convergence as Reynolds number is higher and higher.

1. Incompressible Navier-Stokes Equations

Many mathematicians and engineers are often faced with flow problems to solve, which is governed by so called Navier-Stokes equations. In many flows, the incompressibility are assumed. Thus, We consider the Navier-Stokes equations for incompressible viscous flow as follows :

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\nabla^2\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega \tag{1.1}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega \tag{1.2}$$

where $\mathbf{u}$, p, $\mathbf{f}$ and ν are the velocity, the pressure, external forces and kinematic viscosity respectively and $\Omega \in \mathbf{R}^N, N = 2,3$ is the flow region.

Since relations (1.1) and (1.2) are not sufficient to define a flow, further conditions have to be given such as the initial condition

$$\mathbf{u}(x,0) = \mathbf{u}_0(x) \quad \text{in } \Omega \text{ with } \nabla\cdot\mathbf{u}_0 = 0 \tag{1.3}$$

and the boubdary condition

$$\mathbf{u} = \mathbf{g} \quad \text{on } , (= \partial\Omega) \text{ with } \int_\Gamma \mathbf{g}\cdot\mathbf{n}\, d, = 0. \tag{1.4}$$

where $\mathbf{n}$ denotes the unit outward normal vector to , . The Variety of boundary conditions are described in Glowinski [?]G and Pironneau [2].

1991 *Mathematics Subject Classification.* 35Q30, 65N30, 76D05.
Key words and phrases. Incompressible Navier-Stokes Equations, Finite Element Methods, Uzawa Algorithm, Upwind Scheme.

For steady case, the term with time derivative is removed in (1.1) and the initial condition (1.3) is not needed. In R. Glowinski and O. Pironneau [4], the operator splitting method,say $\theta-$ scheme, was applied to unsteady Navier-Stokes equations, which is the approach in Bristeau and Glowinski et al [5]. In this survey, We consider steady incompressible Navier-Stokes flow only.

Using the finite element approximation, there arise four major difficulties on solving incompressible Navier-Stokes flow defined by (1.1),(1.2),(1.3) and (1.4) :

(1) how to triangulate the given complex domain
(2) how to choose elements for the approximation of $\mathbf{u}$ and the approximation of p to guarantee the existence of finite elemet solution pair $(\mathbf{u}, p)$.(mixed Galerkin)
(3) how to satisfy the incompressibility constraint in (1.2)
(4) how to deal with the nonlinearity in (1.1)

Triangulation problem (1) is categoried as preprocessor for finite element methods. One of the methods is Delaunay triangulation. Mesh adaptation for each problem are modern trend of mesh generation. Recent work for adaptive triangulation has been reported in H. Borouchaki and P. J. Frey [3].

The **mixed Galerkin finite element** formulation for steady incompressible Navier-Stokes flow is written as follows :

(1.5)
$$\mathbf{u} - \mathbf{u}_0 \in V_h \subset [H_0^1(\Omega_h)]^N, p_h \in M_h \subset L_0^2(\Omega_h)$$
(1.6)
$$\nu \int_{\Omega_h} \nabla \mathbf{u}_h : \nabla \mathbf{w}_h \, dx + \int_{\Omega_h} ((\mathbf{u}_h \cdot \nabla)\mathbf{u}_h) \cdot \mathbf{w}_h \, dx$$
(1.7)
$$+ \frac{1}{2} \int_{\Omega_h} (\nabla \cdot \mathbf{u}_h)(\mathbf{u}_h \cdot \mathbf{w}_h), dx - \int_{\Omega_h} p_h(\nabla \cdot \mathbf{w}_h) \, dx = \int_{\Omega_h} \mathbf{f} \cdot \mathbf{w}_h \, dx, \quad \forall \mathbf{w}_h \in V_h$$
(1.8)
$$\int_{\Omega_h} (\nabla \cdot \mathbf{u}_h) q_h \, dx = 0, \quad \forall q_h \in M_h.$$
where V_h and M_h are suitable approximation spaces and $N = 2, 3$.

REMARK The third term in left-hand side of (1.6) plays a roll in ensuring the coercivity so that it may be commonly used in CFD.

2. Element Types for Navier-Stokes Flow Computation

There are two types of approximation in pressure :

(1) discontinuous pressure approximation : allow discontinuity at each interface of elements
(2) continuous pressure approximation : continuous at each interface of elements

Unfortunately, when using mixed Galerkin formulation, all pair of approximation spaces V_h and M_h cannot be chosen arbitrarily since the viscous flow problem is saddle-point problem basically.[6]

The LBB condition or inf-sup condition for the discrete spaces V_h and M_h is sufficient condition to guarantee the existence of FEM solution $(u_h, p_h) \in V_h \times M_h$ satisfying (1.5),(1.6) and (1.8). The condition is that there exist $\beta > 0$ independent of mesh size h such that

$$\sup_{0 \neq \mathbf{v}_h \in V_h} \frac{(\nabla \cdot \mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{1,\Omega_h}} \geq \beta \|p_h\|_{0,\Omega_h}, \quad \forall p_h \in M_h. \tag{2.9}$$

The most popular triangular elements stable are listed below as shown in Figure 1:

- $\mathcal{P}_1 - P_0$ (Fortin, Bernardi & Raugel)[7]
- $P_1^+ - P1$ (Mini Element) [8]
- $P_{k+1} - P_k, k \geq 1$ (Hood-Taylor Element)[9]
- Bercovier-Pironneau Element [10]

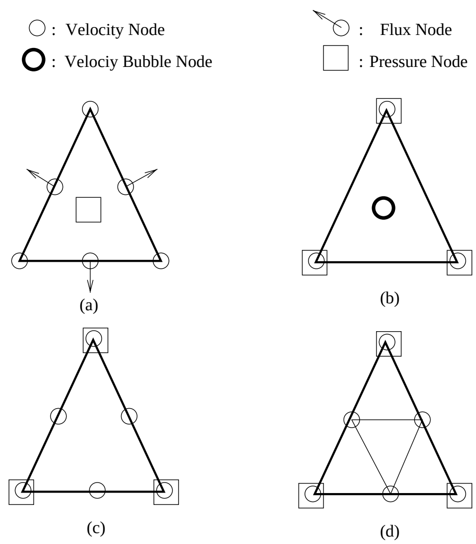

FIGURE 1. Triangular elements satisfying LBB stability condition : (a) $\mathcal{P}_1 - P_0$,(b) $P_1^+ - P_1$, (c) $P_2 - P_1$ and (d) Bercobier-Pironneau

Recent works for quadrilateral finite element stability have been reported in F. H. Bertrand et al and Wen Bai for quadrilateral mini element. [11, 12]

3. Incompressibility : Divergence Free Condition

Let us introduce the generalized Stokes problem(GSP) which is needed to solve numerically incompressible fluid dynamics [13]:

$$\alpha \mathbf{u} - \nu \nabla^2 \mathbf{u} + \nabla P = \mathbf{f} \quad \text{in } \Omega \tag{3.10}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \tag{3.11}$$

$$\mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega \tag{3.12}$$

where $\alpha \geq 0$ and $\nu > 0$

Almost every scheme has used GSP. The incompressibility makes trouble too in here as well as incompressible Navier-Stokes flow.

There are direct iterative solvers for GSP :

- Conjugate Gradient Method(CG)
- Incomplete Cholesky Conjugate Gradient Method(ICCG)

The popular method between velocity divergence and pressure of incompressible flows is so-called **Uzawa Alogorithm** which is derived from saddle-point approach of GSP.

⟨**Uzawa**⟩

$$P_n \xrightarrow{U_1^{opt}} \mathbf{u}_n \xrightarrow{U_2^{opt}} \nabla\cdot\mathbf{u}_n \xrightarrow{U_3^{cor}} P_{n+1} \tag{3.13}$$

- $U_1^{opt} : P \to u$ such that

$$\alpha\mathbf{u} - \nu\nabla^2\mathbf{u} = -\nabla P + \mathbf{f}$$

- $U_2^{opt} : \mathbf{u} \to \nabla\cdot\mathbf{u}$
- $U_3^{cor} : \phi \to P - \rho\phi$

The Uzawa opreator $U^{opt} : L_0^2(\Omega) \to L_0^2(\Omega)$ can be defined by

$$U^{opt}(P) = U_2^{opt} \circ U_1^{opt}(P). \tag{3.14}$$

⟨**Variants of Uzawa**⟩

- Augmented Lagrangian Method : Modify U_1^{opt} operator
- Preconditioning Uzawa Method : Modify U_3^{cor} operator

The major interests in Preconditioning is to find how to approximate the Uzawa operator U^{opt} in order to calculate its inverse easily.

$$U_3^{cor}(\phi) = P - \rho[U^{opt}]^{-1}(\phi) \tag{3.15}$$

If C is a chosen discrete Uzawa operator, its preconditioning Uzawa is written as follows

$$P_{n+1} = P_n - \rho\, C^{-1}\nabla\cdot\mathbf{u}_n \tag{3.16}$$

⟨ **Several Preconditioners** ⟩

- $[U^{opt}]^{-1} \approx \nabla\cdot[-\nabla^2]^{-1}$, Neumann B.C. (Labadie)
- $[U^{opt}]^{-1} \approx \nu\mathbf{I}^{-1} - \alpha(\nabla^2)^{-1}$, Neumann B.C. [13]
- $[U^{opt}]^{-1} \approx$ modified Arrow-Hurwics algorithm [14]

For Navier-Stokes flow, U_1^{opt} is not symmetric therefore in this case **GCG** or **GMRES** is efficient solver for $[U_1^{opt}]^{-1}$.[15]

4. Nonlinearity : Convection

The incompressible Navier-Stokes flow problems can be solved by iterative scheme essentially as well as any other nonlinear problems. [16] The dimensionless form of incompressible Navier-Stokes equations has the parameter called **Reynolds number** denoted by $Re = \frac{UL}{\nu}$.

⟨ **Iterative Schemes** ⟩

- Newton-Raphson Method (low Re flow, $Re = O(1)$)
- Simple Picard Method (intermediate Re , $Re = O(10)$)
- Successive Appromation Method (intermediate Re, $Re = O(10^2)$)
- Upwinding Method (high Re flow , $Re = O(10^4)$)

[Newton-Raphson Method]

- initial guess $\mathbf{u}_h^0$ sufficiently close to the solution
- small iteration for solution
- very small Re

[**Simple Picard Method**]

(4.17)
$$-\frac{1}{Re}\int_{\Omega_h} \nabla \mathbf{u}_h^{n+1} \nabla \mathbf{w}_h \, dx - \int_{\Omega_h} p^{n+1} \nabla\cdot \mathbf{w}_h \, dx$$
(4.18)
$$= \int_{\Omega_h} \mathbf{f}\cdot\mathbf{w}_h \, dx - \int_{\Omega_h} (\mathbf{u}_h^n \cdot \nabla \mathbf{u}_h^n)\cdot \mathbf{w}_h \, dx - \frac{1}{2}\int_{\Omega_h} (\nabla\cdot \mathbf{u}_h^n)(\mathbf{u}_h^n \cdot \mathbf{w}_h) \, dx$$
(4.19)
$$\int_{\Omega_h} q_h \nabla\cdot \mathbf{u}_h^{n+1} \, dx = 0$$

[**Successive Approximation Method**]

$$-\frac{1}{Re}\int_{\Omega_h} \nabla \mathbf{u}_h^{n+1} \nabla \mathbf{w}_h \, dx + \int_{\Omega_h} (\mathbf{u}_h^n \cdot \nabla \mathbf{u}_h^{n+1})\cdot \mathbf{w}_h \, dx \tag{4.20}$$
$$+\frac{1}{2}\int_{\Omega_h} (\nabla\cdot \mathbf{u}_h^n)(\mathbf{u}_h^{n+1} \cdot \mathbf{w}_h) \, dx - \int_{\Omega_h} p^{n+1} \nabla\cdot \mathbf{w}_h \, dx \tag{4.21}$$
$$= \int_{\Omega_h} \mathbf{f}\cdot\mathbf{w}_h \, dx \tag{4.22}$$
$$\int_{\Omega_h} q_h \nabla\cdot \mathbf{u}_h^{n+1} \, dx = 0 \tag{4.23}$$

[**Upwinding Method**]
The aim of upwinding scheme is enhencement of the discrete **ellipticity**.[17, 18] In high Reynolds number flow, upwinding technique has to be applied to the scheme, if not, the convergence cannot be obtained.

$$\mathbf{u}\cdot\nabla\phi(x) \approx \frac{\phi(x)-\phi(y)}{\lambda} \tag{4.24}$$
$$y = x - \lambda \mathbf{u} \quad (\lambda > 0) \tag{4.25}$$

- first order upwinding scheme
- higher order upwinding scheme(third order etc.)

Recent works are done mainly with higer order schemes.[17, 18]

For **unsteady flow**, one of modern methods is **the operator splitting method** involving in time discretization. Usually the $\theta-$ scheme is chosen to solve unsteady Navier-Stokes flow. [4]

REMARKS for Least-Square Formulation

The least-square FEM requires **smoothness** of the finite element spaces so that this approach has received relatively little attention until recently. But by introducing new flux variables, the smoothness constraint can be relaxed. In flow problems, the LBB condition ensuring stability in Galerkin method can is relaxed in the least-square method,i.e., a least-square finite element model using **equal-order basis functions** for velocity and pressure has been successfully developed.[19, 20, 21]

5. Examples

We calculate the cavity flow and the duct flow with slit as examples of 2D incompressible Navier-Stokes flow. The method used here is Hood Taylor and successive approximation. The results are shown in Figure 2,3 as below.

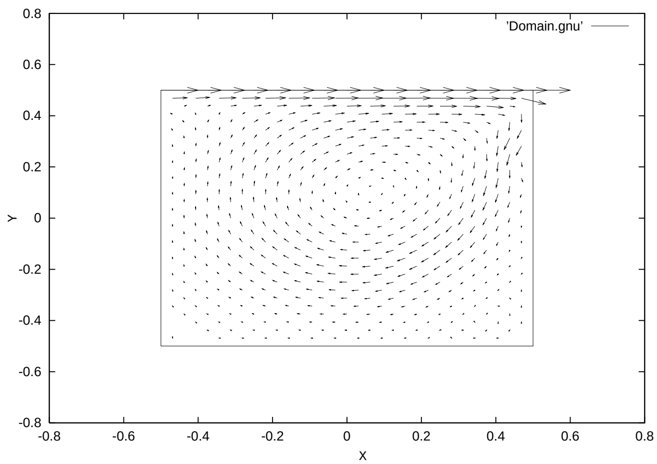

FIGURE 2. Cavity Flow : 256 elements, $Re = 600$

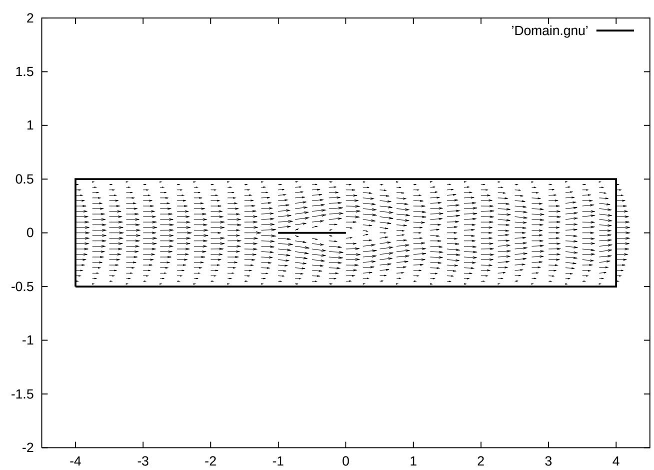

FIGURE 3. Duct Flow with Horizontal Slit : 320 elements, $Re = 600$

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