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AN INTRODUCTION TO THE SINGULARITIES OF HARMONIC MAPS

HEAYONG SHIN

ABSTRACT. Harmonic maps in general are not necessarily smooth. There can be a nonempty singular set at which a harmonic map is not continuous. The nature and structure of such singular sets are not understood very well but in recent years significant developments have been made in this subject. In this note, we give a brief introductory survey on the theory of singularities of minimizing harmonic maps.

1. INTRODUCTION

Harmonic maps between Riemannian Manifold M and N are the critical points u of energy functional

$$E(u) = \int_M |\nabla u|^2 dV$$

where dV is the volume element of M and $|\nabla u|^2$ is the square of the norm of the differential of $u : M \to N$. In local coordinate, $|\nabla u|^2 = \Sigma g^{ij} \frac{\partial u^\alpha}{\partial X^i} \frac{\partial u^\beta}{\partial X^j} h_{\alpha\beta}$ where (g_{ij}) and (h_{ij}) are the coefficients of metric tensor for M and N respectively. We assume that N is isometrically embedded in R^k, then the Euler-Lagrange equation for E can be written as

$$-\Delta_M u = A_N(u)(\nabla u, \nabla u) \tag{1}$$

where Δ_M is the Laplace-Beltrami operator on M and $A_N(u)$ is the second fundamental form of N at u.

For the above functional it is natural to consider the Sobolev space $H^{1,2}(M, N) = \{u \in H^{1,2}(M, R^k) | \; u(X) \in N \; a.e.\}$ and a map $u \in H^{1,2}(M, N)$ is said to be (weakly) harmonic if it satisfies (1) in weak sense. Note that this corresponds to the variational equation

$$\left.\frac{d}{dt}\right|_{t=0} E(\Pi_N(u + t\zeta)) = 0$$

where $\zeta \in C_0^\infty(M, R^k)$ and Π_N is the nearest point projection into N .

When $\dim(M) = 1$, the harmonic maps are the geodesics on N and they are harmonic functions when $N = R$. In these cases, we know that the geodesics or harmonic functions are always smooth. But in general, harmonic maps may have discontinuities (called singularities) because of the topological and geometric constraints. As an example, we may consider $g(X) = \frac{X}{|X|} : S^n \to S^{n-1}$ which, in fact, minimizes the energy under the given boundary data. The existence, structure and asymptotic behavior of the singularities of such harmonic maps are considered as important subjects by many mathematicians.

In this note, we will give brief introduction to the development for the theory of singularities of harmonic maps. We will concentrated mainly on the singularities of energy minimizing maps. We do not claim any completeness on this survey and anyone interested is advised to refer more detailed survey by R. Hardt [9]. General references on harmonic maps can be found in the reports of harmonic maps by Eells and Lemaire [5,6] or in the biography [3].

2. Regularity theory of harmonic maps

Under certain conditions, we do have regularity results on harmonic maps. In 1964, J. Eells and J. H. Sampson proved that if N is compact and has nonpositive curvature then there exists a smooth harmonic map in every homotopy class of maps from closed M to N[7]. There are also complete existence and smoothness results of harmonic maps for the case where the image is contained in a small ball [12].

But for the general case, a harmonic map can have singularities and R. Schoen and K. Uhlenbeck has established a fundamental partial regularity theory for the energy minimizing maps in 1983. They proved that

Theorem [15]. *Let $u : M \to N$ be an energy minimizing map and N be compact. Let $S = \{x \in M | x$ is not Hölder continuous in any neighborhood of $x\}$ be the singular set of u. Then the Hausdorff dimension $\dim(S) \leq \dim(M) - 3$, if $\dim(M) > 3$ and S is a discrete set if $\dim(M) = 3$.*

Hölder continuity of u implies smoothness on $M \backslash S$. When $\dim(M) = 1$, it is clear that any weakly harmonic maps are smooth geodesics. When $\dim(M) = 2$, F. Helein proved the regularity for weakly harmonic maps [11].

Now we will consider some of the basic theorems in the work of Schoen-Uhlenbeck which are proved to be valuable in the study of the singularities of harmonic maps. For the simplicity of the notions we will assume that M is a domain Ω in R^n. This assumption do not cause any essential changes in the theory.

Theorem [15] (monotonicity). *If $u : \Omega \to N$ is energy minimizing, $a \in \Omega$, then*

$$\sigma^{2-n} \int_{B_{\sigma(a)}} |\nabla u|^2 dx - \rho^{2-n} \int_{B_{\rho(a)}} |\nabla u|^2 dx = 2 \int_{B_\sigma(a) \backslash B_\rho(a)} |x|^{n-2} |\frac{\partial u}{\partial r}|^2 dx$$

where $B_\rho(a) \subset B_\sigma(a) \subset M$.

Theorem [15] (ϵ-regularity). *There exists $\epsilon(n, N) > 0$ such that if $u : B_1(0) \to N$ is energy minimizing map with $E(u) \leq \epsilon$, then u is Hölder continuous on $B_{\frac{1}{2}}(0)$.*

The monotonicity formula is a direct consequence of the fact that u satisfies the variational condition

$$\left.\frac{d}{dt}\right|_{t=0} E(u \circ \Phi_t) = 0 \tag{2}$$

where Φ_t is a smooth deformation of identity of Ω such that $\{x|\Phi_t(x) \neq x\} \subset\subset \Omega$. We say a weakly harmonic map is stationary if it also satisfies (2). Smooth harmonic maps and energy minimizing harmonic maps are stationary harmonic.

Another important concept is the tangent map u_0 of an energy minimizing map u at some $a \in \Omega$ which is a weak $H^{1,2}$-limit of $u_{r_i}(x) = u(a + r_i x) : B_1(0) \to N$ for some sequence $r_i \to 0$. The existence of the tangent map follows from the monotonicity formula. But in general we do not have uniqueness of the tangent map. When u is energy minimizing, the above convergence is strong and monotonicity formula implies that u_0 is of homogeneous of degree 0 i.e. $\frac{\partial u_0}{\partial r} = 0$. In fact, the tangent map u_0 itself is an energy minimizing map. Any minimizing map $u_0 : R^n \to N$ which is homogeneous of degree 0 is called a minimizing tangent map(MTM).

From the monotonicity theorem and ϵ-regularity theorem, it is easy to conclude that the singular set S is

$$S = \{x \in \Omega | \lim_{\rho \to 0} \rho^{2-n} \int_{B_{\rho(x)}} |\nabla u|^2 dx \neq 0\}$$

and by a simple covering argument one can check that (n-2)-dimensional Hausdorff measure $\mathcal{H}^{n-2}(S) = 0$. Schoen-Uhlenbeck have used some dimension reduction argument and the concept of the tangent map to prove the better estimate $\dim_{\mathcal{H}}(S) = n - 3$.

Later, Bethuel proved the partial regularity $\mathcal{H}^{n-2}(S) = 0$ for the stationary harmonic maps [1]. For general weakly harmonic maps, we do not have such partial regularity, T. Riviere gave an example of an everywhere discontinuous weakly harmonic maps from B^3 to S^2 [14].

3. Singularities of minimizing maps

Since the tangent map at a singular point of an energy minimizing map u is a limit of a blow up sequence u_{r_i} of u, one can expect to obtain useful informations about the structure and asymptotic behavior of the singularities of energy minimizing map by considering the possible tangent maps on the given situation. For example

Theorem [15]. *Suppose there is no nonconstant MTM from R^j to N for $3 \leq j \leq l$. Then any energy minimizing map $u : M \to N$ has singular set S with* $\dim S \leq \dim(M) - l - 1$.

But, unfortunately we do have only a few examples of minimizing tangent maps and we are not anyway near the complete list of minimizing tangent maps. Among the examples, we have $g(x) = \frac{x}{|x|} : B^n \to S^{n-1}$ which is proved to be energy minimizing by Brezis-Coron-Lieb and others [2,4,13]. Also Colon and Gulliver have proved that the map $g(x) = H(\frac{x}{|x|}) : B^4 \to S^2$, where $H : S^3 \to S^2$ is the Hopf map, is a MTM [4]. A minimizing tangent map with singularities of dimension ≥ 1 can be given as $g(x) = \frac{\Pi(x)}{|\Pi(x)|} : B^n \to S^{j-1}$ where $\Pi(x_1, \ldots, x_n) = (x_i, \ldots, x_j)$, $n > j \geq 3$. Another example of MTM having finite number of lines intersecting at $0 \in B^4$ as singular set can be constructed by taking the target manifold as $S^2 \times S^2 \times \cdots \times S^2$.

The lowest dimension where a singularity of minimizing harmonic map can occur is when $\dim(M) = 3$ and $\dim(N) = 2$. In this dimension, any non constant MTM is of the form $u(x) = \omega(\frac{x}{|x|})$ where ω is conformal (or anticonformal) and N must be topologically S^2 or RP^2. When N is the standard sphere S^2 we have the following uniqueness of minimizing tangent map.

Theorem[2]. $u(x) = \omega(\frac{x}{|x|}) : B^3 \to S^2$ *is energy minimizing if and only if* ω *is a rotation of* S^2.

For general topological sphere N we only have the existence and uniqueness of MTM for $\omega(\frac{x}{|x|})$ where $\omega : S^2 \to N$ has topological degree ± 1 [17]. For the above theorem B-C-L has proved that for $\omega(\frac{x}{|x|})$ to be minimizer $\omega : S^2 \to S^2$ must satisfy balance condition $\int_{S^2} |\nabla \omega(x)|^2 x dA(x) = 0$, and for ω with $\deg(\omega) = \pm 1$, the condition is satisfied only when ω is a rotation. The nonexistence of minimizing $\omega(\frac{x}{|x|})$ with $|\deg(\omega)| \geq 2$ was proved by direct comparison of energy with a map which splits the singularity. For general sphere N, same balance condition is valid and one can prove the uniqueness and existence of degree 1 conformal map $\omega : S^2 \to N$, for which $\omega(\frac{x}{|x|})$ is MTM. But the existence or nonexistence of MTM with $|\deg(\omega)| \geq 2$ is not known.

For higher dimension, classifying MTM seems difficult and we do not know much in this direction. We only have few examples MTM as mentioned above.

Another difficulty in the use of MTM for the investigation of singular set of energy minimizing map occurs from the fact that the tangent map is not necessarily unique in general. B. White gave an example of nonuniqueness of tangent map at an isolated singularity of a harmonic map into a smooth (nonanalytic) manifold N [21]. When N is analytic, L. Simon proved uniqueness of tangent map at isolated singularities.

Theorem [18]. $u : M \to N$ *is an energy minimizing map,* N *is real analytic and* $a \in M \cap sing u$*. If* u_0 *is a tangent map to* u *at* a *with* $sing(u_0) = \{0\}$*, then* u_0 *is the unique tangent map at* a*.*

When $\dim(M) \geq 4$, the singular set of energy minimizing map can have dimension ≥ 1. The structures of the singularity in this case are not fully understood yet. But there are some important results in this direction. For the special case where $M = B^4$ and $N = S^2$, the shape of the singular set can be analysed, as following.

Theorem [10]. *If* $u : B^4 \to S^2$ *is an energy minimizing map with smooth boundary data, then singular set consists of a finite set and finitely many Hölder continuous closed curves with only finitely many crossings.*

In this case, it is unknown whether there are crossings and whether the singular set has finite $\mathcal{H}^1$-measure. Recently L. Simon proved following beautiful theorem for general real analytic target manifold.

Theorem [19]. *If* $u : M \to N$ *is energy minimizing and* N *is compact, real analytic manifold, then for any closed ball* $B \subset M$, $sing(u) \cap B$ *is the union of finite pairwise disjoint collection of locally* $(n-3)$*-rectifiable locally compact subsets.*

In this note we introduced a short survey on the theory of singularities of harmonic maps which is concentrated mainly on that of energy minimizing maps. There are many other important results on this subject which should be mentioned but omitted.

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