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EXTREMAL KÄHLER METRICS ON RULED SURFACES

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ABSTRACT. In this short note we give a brief survey of recent works on extremal Kähler metrics on (not necessarily minimal) ruled surfaces. A minimal ruled surface is a compact complex 2-dimensional manifold which is the total space of a holomorphic fiber bundle with fiber $\mathbb{CP}_1$ over a connected holomorphic curve, whereas a non-minimal ruled surface is a minimal ruled surface with some points blown up so that it admits rational curves with self intersection -1. We discuss mainly the existence and uniqueness of extremal Kähler metrics defined by E. Calabi on ruled surfaces.

1. **Introduction**

This note is for a 40 minutes lecture in the ICMS mathematics conference held at KAIST, Korea in the summer of 1998.

We call a Riemannian metric on a compact smooth oriented four-dimensional manifold M to be $\mathcal{R}$-critical if it is a critical point of the Riemannian functional $\mathcal{R}(g) = \int_M |R_g|_g^2 dv_g$ defined on the space of all smooth Riemannian metrics g, where R_g is the full Riemannian curvature tensor of g .

While it is not easy to understand general $\mathcal{R}$-critical metrics, $\mathcal{R}$-critical metrics which are Kähler are characterized to be precisely Kähler Einstein or Kähler metrics of zero scalar curvature [6]. These two classes of metrics are quite special and not all Kähler surfaces can admit one of these. Indeed, a Kähler Einstein metric of positive ricci curvature exists on blow-ups of $\mathbb{CP}_2$ at three to eight points, zero ricci curvature on finite quotients of complex tori or K3 surfaces and negative ricci curvature on general type surfaces with negative first chern class [14, 15]. And Kähler metrics of zero scalar curvature exist on some ruled surfaces, to be explained in section 2.

So recall that in [3] E. Calabi has defined a Kähler metric on a compact complex manifold to be extremal Kähler if it is a critical point of the $\mathcal{R}(g)$ functional defined on the subspace of Kähler metrics with their associated Kähler forms in a fixed cohomology class. The extremal Kähler metrics are of constant scalar curvature if there exist no nontrivial holomorphic vector fields, which is the case for general compact complex surfaces. And this is one of the reasons why extremal Kähler metrics with non-constant scalar curvature on ruled surfaces are interesting. Recall that compact complex surfaces are roughly classified by Kodaira dimension. If a Kähler surface has Kodaira dimension 2 then it is a general type surface and if Kodaira dimension 1, it is an elliptic surface. If Kodaira dimension is 0, then it is (a quotient of) Torus, K3 surface or a hyperelliptic surface. A Kähler surface of the remaining Kodaira dimension $-\infty$ is either $\mathbb{CP}_2$ or a ruled surface.

For above reason or others, we are interested to study extremal Kähler metrics on ruled surfaces. There are two basic questions; their existence and uniqueness. First we ask which ruled surface and which Kähler class can admit an extremal

Kähler metric? Second if one such a metric exists on a fixed ruled surface is it unique up to holomorphic isometries in the Kähler class? In section2 we explain recent results on Kähler metrics with zero scalar curvature and in section 3 on extremal Kähler metrics with non-constant scalar curvature on ruled surfaces.

2. Kähler metrics with zero scalar curvature

There are strong restrictions for a compact complex surface to admit a Kähler metric with zero scalar curvature. According to [16] such a surface has Kodaira dimension $-\infty$ or is covered by a complex torus or K3 surface. As any Kähler metric with zero scalar curvature on a complex torus or a K3 surface should be indeed Kähler ricci-flat by simple Riemannian geometric argument, we put aside this case. A complex surface of Kodaira dimension $-\infty$ is biholomorphic to $\mathbb{CP}_2$ or a ruled surface. A ruled surface can be divided into three classes according as the genus $g(C)$ of its base curve C is greater than, equal to or less than 1. In order to admit a Kähler metric with zero scalar curvature, the square of the first chern class, ${c_1}^2$, has to be negative. Therefore if $g(C)$ is greater than 1, then the ruled surface has to be blown up at least at 9 points and if $g(C)$ is equal to 1, then it has to blown up at least once. Similarly $\mathbb{CP}_2$ can not have such a metric.

Topological structures of ruled surfaces are simple; for minimal ones there are only two distinct underlying smooth structures associated to a genus number of the base curve and for non-minimal ones, only the number of blowing ups are to be counted additionally. But on a fixed smooth (C^∞) ruled surface, there are infinitely many complex structures determined by several data, i.e. a complex structure of its base curve, the bundle structure, and the way to blow up points.

Our first basic question is on existence; on a smooth ruled surface which complex structures and which Kähler classes admit Kähler metrics with zero scalar curvature? Here, a Kähler class is an element of the second cohomology group $H^2(M, R)$ which can be represented by a Kähler form on M.

To illustrate some points regarding existence, let us look at a simple case; we can characterize minimal ruled surfaces admitting Kähler metrics with zero scalar curvature as follows. According to [2] or other argument, it holds that the universal cover $\tilde{S}$ of such a ruled surface S with induced metric is holomorphically isometric to the product $\Delta \times \mathbb{CP}_1$, where Δ is the Poincare disk and that S is a unitary flat $\mathbb{CP}_1$ bundle over the base curve C. That is, we have a representation of $\pi_1(C)$ into $SU(2)$. If E is the associated flat unitary bundle we have that $S \cong \mathbb{P}(E)$ holomorphically. As E is an Einstein-Hermitian vector bundle, E is *quasi-stable* [9, 13], i.e. E is semi-stable and isomorphic to a direct sum of stable Hermitian sub-bundles. This description is a complete answer to the existence question of Kähler metrics with zero scalar curvature on minimal ruled surfaces. For instance, $\mathbb{P}(\mathcal{L} \oplus \mathcal{O}) \to C$ with $deg(\mathcal{L}) \neq 0$ cannot admit such a metric, where $\mathcal{L}$ is a holomorphic line bundle over C and $\mathcal{O}$ is the trivial line bundle over C.

C. LeBrun and M. Singer have worked on non-minimal ruled surfaces M with nontrivial holomorphic vector fields admitting Kähler metrics with zero scalar curvature [10]. According to their work, M can be described as the following; for some holomorphic line bundle $\mathcal{L} \to C$ over a compact complex curve C, M is obtained from the minimal ruled surface $\mathbb{P}(\mathcal{L} \oplus \mathcal{O}) \to C$ by blowing up points along the zero section of $\mathcal{L} \in \mathbb{P}(\mathcal{L} \oplus \mathcal{O})$. Futhermore the Kähler classes on above ruled surfaces were determined in terms of computed Futaki invariant values, see also [8]. LeBrun and Singer have shown not only above necessary condition but also constructed Kähler metrics with zero scalar curvature on some of the above ruled surfaces, i.e. those which admit a *semi-free* isometric circle action. In [7, 8], one can find constructions on ruled surfaces with a non-semi-free circle action. These constructions are still far from completion; in [8] we have constructed metrics on blow-ups of $\mathbb{CP}_1 \times \mathbb{CP}_1$ at 13 or more points, but many questions are open, e.g. if a blow-up of 9 to 12 points of $\mathbb{CP}_1 \times \mathbb{CP}_1$ can admit such a metric or if a certain distribution of blowing up 13 or more points (so that the blown up surface has a nontrivial holomorphic vector field) can admit one.

For the generic case, i.e. non-minimal ruled surfaces with no nontrivial holomorphic vector fields, we have many examples from constructions [7, 8] and deformations [11], but complete descriptions are still missing. This leaves an interesting open question. One may compare this situation with the Kähler Einstein metric case where complex structures admitting those are precisely identified.

Our second basic question is on uniqueness of Kähler metrics of zero scalar curvature up to homothety in a fixed Kähler class. For Kähler Einstein metrics this uniqueness holds due to the work of Bando and Mabuchi [5]. For ruled surfaces with nontrivial holomorphic vector fields this question is not an issue because LeBrun and Singer's description itself answers positively to the question. So real question remains on other ruled surfaces.

3. Extremal Kähler metrics on ruled surfaces

As explained in the introduction, extremal Kähler metrics with non-constant scalar curvature are meaningful on surfaces with nontrivial holomorphic vector fields, so in particular on ruled surfaces. There are found some interesting examples of non-constant scalar curved extremal Kähler metrics. In fact Calabi has concretely described the first example of extremal Kähler metrics on $\mathbb{CP}_2$ blown up at a point [1]. An interesting feature is that one of these extremal Kähler metrics is globally conformal to a Hermitian Einstein metric found by D. Page. A very interesting related question is whether there exists an extremal Kähler metric on $\mathbb{CP}_2$ blown up at *two* points which is conformal to a Hermitian Einstein metric. Generalizations of Calabi's construction have continued until recently and among them one can refer to [4, 12] for surface case. In [4], she constructed extremal Kähler metrics on $\mathbb{P}(\mathcal{L} \oplus \mathcal{O}) \to C$ with $deg(\mathcal{L}) \neq 0$ and genus of $C \geq 2$. She also showed that if a minimal ruled surface with genus bigger than or equal to two admits an extremal Kähler metric with non-constant scalar curvature, then it is biholomorphic to one of above surfaces.

Let us explain a little more on the non-existence result of [2]. In section 2 we described minimal ruled surfaces which admits Kähler metrics with zero scalar curvature. Bartolomeis and Burns have considered certain non stable, but indecomposable rank 2 holomorphic bundle E over a curve with genus bigger than or equal to 2. Then $\mathbb{P}(E)$ has no nontrivial holomorphic vector field so any extremal Kähler metric should have constant scalar curvature. Choosing a Kähler class so that the scalar curvtuare should equal zero, they showed that E canot be quasi-stable. This implied that the surface does not admit any extremal Kähler metric.

As explained above, quite a few results are made concerning (non)-existence. There are many interesting questions to ask. For instance, one may try to generalize

[4] argument to the cases when the base curve has genus equal to 1 or to the case of general non-minimal ruled surfaces.

The uniqueness question is valid for any extremal Kähler metric, not just for Kähler metrics with zero scalar curvature on ruled surfaces. On some ruled surfaces existence and uniqueness were observed to be interwined. For this readers may refer to [4].

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