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MANIFOLDS OF ALMOST NONNEGATIVE RICCI CURVATURE

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Abstract. In this survey article, we shall describe recent results about topology of Riemannian manifolds of almost non-negative Ricci curvature.

We say a Riemannian manifold (M, g) is of *almost* non-negative Ricci curvature if (M, g) satisfies

$$Ric(M) \cdot diam(M)^2 \geq -\epsilon \qquad (*)$$

for a small positive number $\epsilon > 0$. First we describe the fundamental group of a Riemannian manifold M contained in such a class (*) is almost nilpotent. Furthermore if we assume the lower bound on the volume, $vol(M) \geq v > 0$, then the fundamental group of such a manifold must be almost abelian.

On the other hand, if the first Betti number, $b_1(M)$, is equal to $n = dim(M)$ for a manifold M in (*), then M must be homeomorphic to a torus if $n \neq 3$ and homotopically equivalent to a torus if $n = 3$.

Besides the results stated above, we also present several properties about the topology of manifolds with sectional curvature condition like (*).

§1. Introduction

One of the main topics in Riemannian geometry is to study the relation of geometric features like curvature, diameter and volume with the topology of manifolds.

In this survey article, we shall describe recent results about topology of Riemannian manifolds of *almost* non-negative Ricci curvature. First of all we consider manifolds of non-negative Ricci curvature. The topological characteristics of non-negative Ricci curvature manifolds are the first Betti number and fundamental groups at present. Those two notions are of course related with each other. The first Betti number corresponds to the rank of the quotient space of the fundamental group by its commutator subgroup. That is, by Hurewicz's theorem

$$\pi_1(M)/[\pi_1(M), \pi_1(M)] \cong H_1(M, \mathbf{Z}).$$

The fundamental group of a compact Riemannian manifold acts on its universal Riemannian covering manifold isometrically and this action is properly discontinuous and fixed point free.

Throughout this paper, M is a closed (i.e., compact without boundary) n-dimensional Riemannian manifold unless otherwise stated. We denote by $Ric(M)$ the Ricci curvature of M.

A classical result in this direction is due to Bochner which is stated as

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Theorem 1.1 ([B-Y]). *Suppose $Ric(M) \geq 0$. Then the first Betti number $b_1(M) = rank\, H_1(M,\mathbf{Z})$ is less than or equal to n, and $b_1(M) = n$ if and only if M is diffeomorphic to a flat torus T^n.*

The proof is based on Bochner-Weitzenböck formula for a harmonic 1-form. We may assume M is orientable. Using Bochner-Weitzenböck formula and assumption on curvature, Bochner showed every harmonic 1-form is parallel, i.e., the covariant derivative is zero. Then since parallel form is automatically harmonic, the harmonic 1-forms coincide with the parallel 1-forms. By the Hodge theory ([Rha]), the space of harmonic 1-forms is isomorphic to $H^1(M, \mathbf{R})$. Thus for a compact oriented Riemannian manifold M of non-negative Ricci curvature, its first Betti number $b_1(M) = dim\, H^1(M, \mathbf{R})$ is equal to the dimension of the maximal parallel distribution (i.e., tangent subbundle) on M. In particular, $b_1(M) \leq dim(M)$, with equality obtained by a flat torus.

We now turn to measure the size of the fundamental group of a manifold. Let us recall the definition of the growth of a finitely generated discrete group. Let , be a finitely generated discrete group with generators $\{\gamma_1, \cdots, \gamma_l\}$, $\gamma_i \in$, . Then every element γ of , can be expressed as a word in $\{\gamma_1, \cdots, \gamma_l\}$,

$$\gamma = \gamma_{i_1}^{k_1} \cdots \gamma_{i_m}^{k_m},$$

where $k_j \in \mathbf{Z}$. The *length* of γ is defined by the summation of absolute value of powers,

$$|\gamma| = |k_1| + \cdots + |k_m|.$$

Now define the growth function, $\#U(r)$, as the number of distinct words in , of *minimal* length $\leq r$.

We say , has polynomial growth of order $\leq k$ if there is a positive constant C such that

$$\#U(r) \leq C\, r^k.$$

It is well-known ([Mil]) that this is well-defined, that is, the property of polynomial growth is independent of generating set $\{\gamma_i\}$. Obviously a free abelian group has a polynomial growth. More generally, it is known that a finitely generated discrete nilpotent group has a polynomial growth ([Gr1], [Wo1]). The converse is also true which is proved by Gromov ([Gr1]).

A group , is called *nilpotent* if denoting [, , ,] = , $^{(1)}$, and [, , , $^{(i)}$] = , $^{(i+1)}$ inductively, we have , $^{(k)} = \{1\}$ for some k. And a group , is called *almost* (or *virtually*) *nilpotent* if it contains a nilpotent subgroup of finite index.

Example 1 (Heisenberg group). Let N be the 3-dimensional Heisenberg group and , the integer lattice:

$$N = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbf{R} \right\}, \quad , = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbf{Z} \right\}.$$

Then the quotient space $M = N/$, is a compact 3-dimensional orientable manifold and the fundamental group is , , i.e., $\pi_1(M) =$, . It is easy to compute that

$$[, , ,] = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : c \in \mathbf{Z} \right\} \cong \mathbf{Z}$$

and so

$$, /[, ,] = \mathbf{Z} \oplus \mathbf{Z} \quad \text{and} \quad b_1(M) = 2.$$

It is also known ([Mil]) that the growth is

$$growth(,) = 4 > dim(M).$$

□

The following result is due to J. Milnor.

Theorem 1.2 ([Mil])**.** *Suppose $Ric(M) \geq 0$. Then $\pi_1(M)$ has a polynomial growth of order $\leq n$.*

Proof. Since M is compact, the funndamental group of M is finitely generated. In fact, using Lebesgue covering property or fundamental domain on universal cover $\widetilde{M}$ of M, one can see that there exists a compact subset K of $\widetilde{M}$ such that

$$\bigcup_{\gamma \in \pi_1(M)} \gamma(K) = \widetilde{M}$$

and this covering is locally finite. This implies that for any $D > 0$, the set

$$S = \{ \gamma \in \pi_1(M) \, : \, d(K, \gamma(K)) < D \}$$

is finite and one can also show $\pi_1(M)$ is generated by S. Here d denotes the Riemannian distance.

Recall now the fundamental group acts isometrically on the universal Riemannian cover $(\widetilde{M}, \widetilde{g})$. Take a point $x_o \in \widetilde{M}$. Using the fundamental domain (Dirichlet domain), one can find $a > 0$ such that the balls $B(\gamma(x_o), a)$ are pairwise disjoint. Take a finite system $S = \{\gamma_i\}$ as above with $D > diam(M)$ and set

$$C = \max \{ d(x_o, \gamma_i(x_o)) \, : \, \gamma_i \in S \}.$$

Now if $\gamma \in \pi_1(M)$ can be represented as a word of length not greater than r with respect to the γ_i, clearly

$$d(x_o, \gamma(x_o)) \leq Cr.$$

Taking such all $\gamma's$, one obtains $\#U(r)$ disjoint balls $B(\gamma(x_o), a)$, such that

$$B(\gamma(x_o), a) \subset B(x_o, Cr + a).$$

Therefore

$$\#U(r) \leq \frac{vol(B(x_o, Cr + a))}{vol(B(x_o, a))} \leq C_M (Cr + a)^n.$$

The last inequality follows from the Bishop volume comparison theorem ([B-C]).□

So the compact 3-dimensional nilmanifold $M = N/,$ in Example 1 does not admit a metric of non-negative Ricci curvature.

In [C-G], Cheeger and Gromoll generalized this result as follows.

Theorem 1.3 ([C-G])**.** *If $Ric(M) \geq 0$, then $\pi_1(M)$ contains a finite index free abelian subgroup of rank $\leq n$.*

The proof for Theorem 1.3 is based on the splitting theorem which is also proved by Cheeger and Gromoll ([C-G]). The universal Riemannian covering $\widetilde{M}$ of M splits isometrically into $\mathbf{R}^k \times M_o$, where M_o is compact and no contains a line. Moreover the isometry group of $\widetilde{M}$ splits into product type too. Finally using Bieberbach theorem ([Wo2]), the proof is completed.

§2. Manifolds of Almost non-negative curvature

Looking at results in section 1, one might ask how much the manifold M could admit negative constant. In other words, if $Ric(M) \geq -1/2$, then does the same reslut as in section 1 still hold for $\pi_1(M)$ or $b_1(M)$. The answer is definitely "NO". Because if it does, by rescaling Riemannian metric, every manifold satisfies such a property for $\pi_1(M)$ or $b_1(M)$ as in section 1, but this should be wrong. So there must be another geometric condition to handle such a problem. The diameter and volume of manifolds are typical geometric constraints to work this kind of porblem. If, togethe with an upper bound on diameter, the curvature of a metric admits a small negative constant, then almost all properties that hold for manifolds of non-negative curvature are still satisfied.

Keeping this in mind, the following is maybe the first result which is due to M. Gromov. The diameter of a Riemanian manifold, (M, g), is denoted by $diam(M)$.

Theorem 2.1 ([G-L-P]). *Given integer n, there exists a positive number $\epsilon = \epsilon(n)$ such that if (M, g) is a closed Riemannian n-manifold satisfying $Ric(M) \cdot diam(M)^2 \geq -\epsilon$, then $b_1(M) \leq n$.*

Note that the condition $Ric(M) \cdot diam(M)^2$ is scale invariant. We would also like to remark that the number ϵ is usually small but it is not known expicitly. For a simple proof, see also [Col].

We say a Riemannian manifold (M, g) is of *almost* non-negative Ricci curvature if (M, g) satisfies

$$Ric(M) \cdot diam(M)^2 \geq -\epsilon$$

for a small positive number $\epsilon > 0$. We also say a Riemannian manifold (M, g) is of *almost* non-negative sectional curvature if (M, g) satisfies

$$K_M \cdot diam(M)^2 \geq -\epsilon$$

for a small positive number $\epsilon > 0$.

Example 2. (Continued) Typical examples of manifolds of almost non-negative curvature are nilmanifolds. In Example 1, since N is a nilpotent Lie group, M admits a left invariant Riemannian metric. In fact, with respect to the standard coordinate x, y, z, the metric $g = dx^2 + dy^2 + (dz - xdy)^2$ is a left-invariant metric with sectional curvature $-1 \leq K_g \leq 1$. For any $\epsilon > 0$ define the left-invariant metric g_ϵ on M as follows: For

$$\begin{pmatrix} 0 & u & w \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix} \in T_eN, \quad \text{define} \quad \left\| \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right\|^2 = u^2 + v^2 + \epsilon^2 w^2,$$

where e is the identity element of N. Then the sectional curvature and diameter of g_ϵ satisfy $|K_{g_\epsilon}| \leq 24\epsilon^2$ and $diam(M, g_\epsilon) \leq 2$, respectively. One can also see easily that (M, g_ϵ) converges to a flat 2-torus T^2 with respect to the Gromov-Hausdorff topology.

The following results are important consequences in our concern. Of course there are many other results which are also valuable, but we do not mention those here and turn to references.

Theorem 2.2 ([Gr2]). *There exists a positive number $\epsilon = \epsilon(n)$ such that if*

$$|K_M| diam(M)^2 < \epsilon,$$

then M is diffeomorphic to an infranilmanifold.

A manifold M is called an *infrmanifold* if a finite covering space of M is a quotient of a simply connected nilpotent Lie group by its lattice.

The following two theorems are generalization of Theorem 2.2 in some sense.

Theorem 2.3 ([Ya2]). *There exists a positive number $\epsilon = \epsilon(n)$ such that if*

$$K_M \cdot diam(M)^2 > -\epsilon,$$

then a finite cover of M fibers over $b_1(M)$-dimensional torus. Moreover in the maximal case $b_1(M) = n$, M is diffeomorphic to a torus

Theorem 2.4 ([F-Y]). *There exists a positive number $\epsilon = \epsilon(n)$ such that if*

$$K_M \cdot diam(M)^2 > -\epsilon,$$

then $\pi_1(M)$ is almost nilpotent.

Next, we state a theorem due to G. Wei which is a generalization of Theorem 1.2.

Theorem 2.5 ([Wei]). *Given n and $D, v > 0$, there exists a positive number $\epsilon = \epsilon(n, D, v)$ such that if*

$$Ric(M) > -\epsilon, \quad dima(M) \leq D, \quad vol(M) > v \tag{+}$$

then $\pi_1(M)$ has polynomial growth of order $\leq n$.

Corollary 2.6 ([F-Y]). *Given n and $D, v > 0$, there exists a positive number $\epsilon = \epsilon(n, D, v)$ such that if*

$$K_M > -\epsilon, \quad dima(M) \leq D, \quad vol(M) > v,$$

then $\pi_1(M)$ is almost abelian, i.e, it contains a finite index free abelian subgroup of rank $\leq n$.

In [K-Y1], using quite different method from ususal, the author and Kim proved the following result related with scalar curvature in 4-dimension case.

Theorem 2.7 ([K-Y1]). *There is a positive number $\epsilon > 0$ such that if (M^4, g) is a closed Riemannian 4-manifold satisfying*

$$|K_M| \cdot diam(M, g)^2 \leq \epsilon,$$

then a finite cover of M is diffeomorphic to T^4 or M does not have zero scalar curvature metrics.

§3. About Proofs

We are now going to discuss the proofs for theorems stated in Section 2 above. To prove those theorems, we need some important notions about structure of Riemannian manifolds. Those are *convergence theorem* with respect to the Gromov-Hausdorff topology, *splitting phenomina* for the universal covering of a given manifold, *volume comparison theorem* and good choice of generators for the fundamental group of a manifold.

The Gromov-Hausdorff topology is a generalization of the classical Hausdorff distance between subsets of a fixed metric space ([G-L-P]).

Definition. A (not necessarily continuous) map $f : X \to Y$ between metric spaces is called an ϵ-*Hausdorff approximation* if

1. $|d(f(x), f(y)) - d(x, y)| < \epsilon$ for all $x, y \in X$,
2. The ϵ-neighborhood of $f(X)$ covers Y.

Then the *Hausdorff distance* $d_{GH}(X, Y)$ is defined as the infimum of ϵ such that there exist ϵ-Hausdorff approximations from X to Y and Y to X.

For unbounded spaces, the notion of *pointed Gromov-Hausdorff distance* is effective. By a pointed metric space we mean a pair (X, p) of a metric space X and a point $p \in X$.

Definition. Let (X_i, p_i) and (X, p) be pointed metric spaces. We say (X_i, p_i) converges to (X, p) in the *pointed Gromov-Hausdorff distance* if for any $r > 0$ and any sequence of positive real numbers $\epsilon_i \to 0$, the closed balls $\overline{B}(p_i, r + \epsilon_i)$ in X_i converges to the closed ball $\overline{B}(p, r)$ in X in the Gromov-Hausdorff distance.

As in compact case, we can define the notion of the pointed Gromov-Hausdorff distance by using Hausdorff approximation.

Definition. For pointed metric spaces (X, p) and (Y, q), the pointed Gromov-Hausdorff distance $d_{p.GH}((X, p), (Y, q))$ is defined as the infimum of $\epsilon > 0$ such that there exist ϵ-Hausdorff approximations $f : B^X(p, \frac{1}{\epsilon}) \to B^Y(q, \frac{1}{\epsilon} + \epsilon)$ and $g : B^Y(q, \frac{1}{\epsilon}) \to B^X(p, \frac{1}{\epsilon} + \epsilon)$ between metric balls with $f(p) = q$ and $g(q) = p$.

For more details, one can refer [G-L-P], [Fuk] or [C-Y] and references are therein.

The reason that the Gromov-Hausdorff distance is useful is the following theorem due to Gromov.

Theorem A (Gromov Precompactness Theorem). *For given $D > 0, k$ and $n \geq 2$,*

1. *the space of all compact Riemannian n-manifolds (M^n, g) satisfying*

$$Ric(M) \geq (n-1)k, \quad diam(M) \leq D \tag{3.1}$$

   is precompact in the Gromov-Hausdorff topology.
2. *the space of all pointed complete Riemannian n-manifolds (M^n, g) satisfying $Ric(M) \geq (n-1)k$ is precompact in the pointed Gromov-Hausdorff topology.*

This implies that if (M_i, g_i) is a sequence of Riemannian n-manifolds satisfying (3.1), then there exists a metric space X such that M_i converges to X in the Gromov-Hausdorff topology. One can also prove that X must be compact in this case. In fact, X is a *length space* and one has $diam(X) \leq D$. Note that k might be also negative number. In case pointed complete Riemannian n-manifolds satisfying $Ric(M) \geq (n-1)k$, the same property holds.

Definition. Let (X, d) be a metric space. We define the *distance of length* on X by

$$d_l(x,y) = \inf\Big\{ l(\gamma) : \gamma(0) = x,\ \gamma(1) = y \Big\},$$

where the infimum is taken over all continuous curves in X joining x and y.

In general, there is no reason for that $d_l = d$ for a given metric space (X, d). In fact, the topologies induced by d and d_l may be different (cf. [G-L-P]).

Definition. A metric space (X, d) is called a *length space* (or *inner metric space*) if $d = d_l$.

As mentioned above, the universal Riemannian covering of a compact Riemannian manifold of nonnegative Ricci curvature satisfies the splitting phenomina due to [C-G]. For complete Alexandrov space with curvature ≥ 0, the same property holds. An Alexandrov space with curvature $\geq k$ means by definition a length space (and so automatically metric space) satisfying the toponogov's triangle comparison theorem (see [C-E]).

Theorem B (Splitting Theorem). *Let X be a complete Alexandrov space with curvature ≥ 0. If X contains a line, then it is isometric to a product $X_o \times \mathbf{R}$.*

The following volume comparison theorems are due to Bishop and Gromov, respectively.

Theorem C ([B-C]). *Let (M, g) be a complete Riemannian manifold and $B_p(r)$ denotes the geodesic ball in M. If $Ric(M) \geq (n-1)k$, then $vol(B_p(r)) \leq V_k(r)$, where $V_k(r)$ denotes the volume of a ball of radius r in the space form of curvature k.*

Theorem D ([G-L-P]). *If (M, g) is a complete Riemannian manifold with $Ric(M) \geq (n-1)k$, then for any point $p \in M$,*

$$\frac{vol(B_p(r))}{V_k(r)}$$

is nonincreasing, i.e., one has for $r' > r > 0$

$$\frac{vol(B_p(r'))}{V_k(r')} \leq \frac{vol(B_p(r))}{V_k(r)}.$$

In particular, if M is compact with the diameter $diam(M) = D$, then for $r \leq D$ we have

$$\frac{vol(B_p(r))}{V_k(r)} \geq \frac{vol(M)}{V_k(D)}.$$

Arguments to prove those theorems stated in Section 2 are as follows: Suppose the given conclusion does not hold. Then there is a sequence of Riemannian manifolds satisfying given assumptions, but not conclusion. With geometric hypotheses a subsequence of this sequence converges in the Gromov-Hausdorff topology to a metric space. Then study the limit space and try to get a contradiction.

Now let us prove Theorem 2.6.

Proof of Theorem 2.6. Suppose the conclusion in Theorem 2.6 does not hold. Then there is a sequence $\epsilon_i \to 0$ and a sequence of Riemannian manifolds (M_i, g_i) satisfying

$$K_{M_i} \geq -\epsilon_i, \quad diam(M_i) \leq D, \quad vol(M_i) \geq v,$$

but $\pi_1(M_i)$ is not almost abelian. Then convergence theorem (Theorem A) shows M_i subconverges to an Alexandrov space with curvature ≥ 0. Applying Splitting Theorem B, X splits isometrically into $X_o \times \mathbf{R}^k$. From convergence, $\pi_1(M_i) \cong \pi_1(X)$ for i sufficiently large. However from the same argument as Theorem 1.3 with splitting, one can see $\pi_1(X)$ is almost abelian. Hence we get a contradiction.□

We finally discuss about good choice of generators. The argument is similar as in the proof of Theorem 1.2.

Lemma 1 ([G-L-P]). *Let (M, g) be a Riemannian manifold with $diam(M) \leq D$. Then for any point $p \in \widetilde{M}$, there exists a finite system of generators $\{\gamma_1, \cdots, \gamma_N\}$ of $\pi_1(M)$ satisfying*

$$d(\gamma_i(p), p) \leq 2D \qquad (1 \leq i \leq N)$$

and

$$\gamma_i \gamma_j \gamma_k^{-1} = 1.$$

For fixed constants $k, D, v > 0$ and $n \in \mathbf{N}$, suppose (M, g) is a closed Riemannian n-manifold satisfying

$$Ric(M) \geq -(n-1)k^2, \quad diam(M) \leq D, \quad vol(M) \geq v. \tag{3.2}$$

Fix a point $p \in \widetilde{M}$ and let F be the fundamental domain of $\pi_1(M) = ,$:

$$F = \bigcap_{\gamma \in \Gamma} \{x \in \widetilde{M} \,:\, d(p, x) \leq d(\gamma(p), x)\}.$$

One can show that F is isometric to M except a subset of measure zero.

For a $\delta > 0$, we put $N = \#,(\delta) := \#\{\gamma \in \,, \;:\; d(\gamma(p), p) \leq \delta\}$, $,(\delta) = \{\gamma_1, \cdots, \gamma_N\}$ and $\alpha_i = \gamma_i \cdots \gamma_1$ $(1 \leq i \leq N)$. Then we have obviously $d(p, \alpha_i(p)) \leq i\,\delta$ and so

$$\bigcup_{i=1}^{l} \alpha_i(F) \subset B_p(l\delta + D), \;\; (1 \leq l \leq N).$$

Thus by Bishop's volume comparison theorem (Theorem C) together with (3.2)

$$l \cdot v \leq l \cdot vol(M) \leq V_k(l\delta + D), \tag{3.3}$$

where $V_k(r)$ denotes the volume of an r-ball in the space form of constant curvature $-k^2$. Letting $\delta = Dv/V_k(2D)$, we have

$$N \leq \frac{V_k(2D)}{v}. \tag{3.4}$$

Let us now evaluate $\#, (2D) = \#\{\gamma \in ,\ : d(\gamma(p), p) \leq 2D\}$. Let $\{x_j\}$ be a maximal subset of $\{\gamma(p) : \gamma \in , (2D)\}$ such that $d(x_j, x_k) \geq \delta$ for $j \neq k$. Let K be the number of $\{x_j\}$. By Theorem C and Theorem D,

$$K \leq \frac{vol\, B_p(2D + \delta/2)}{vol\, B_p(\delta/2)} \leq \frac{V_k(2D + \delta/2)}{V_k(\delta/2)}.$$

This yields that

$$\#, (2D) \leq K\#, (\delta) \leq \frac{V_k(2D)}{v} \cdot \frac{V_k(2D + \delta/2)}{V_k(\delta/2)}.$$

Hence we proved the following

Theorem E ([And]). *The set of all isomorphism classes of fundamental groups of closed n-manifolds with bounds (3.2) is finite.*

We now prove Theorem 2.5.

Proof of Theorem 2.5. Let M satisfy the bounds (+) for $\epsilon > 0$. From Lemma 1, we can take generators $\gamma_1, \cdots, \gamma_L$ of $\pi_1(M) = ,$ such that $d(\gamma_i(p), p) \leq 2D$. Note that L is uniformly bounded by (3.4). Let $\#U(r)$ be the number of words in , of length $\leq r$ with respect to $\gamma_1, \cdots, \gamma_L$. Similarly to (3.2) we have

$$\#U(r)\, v \leq V_k(2rD + D), \quad (n-1)k^2 = \epsilon. \tag{3.5}$$

If $\#U(r)$ is not of polynomial growth of order $\leq n$ for any sufficiently small ϵ, there exists a sequence $r_i \to \infty$ such that $\#U(r) > i\, r_i$. Since there are only finitely many possibilities for the isomorphism classes of , by Theorem E, one can take r_i independent of M. On the other hand, by (3.5) for any large r one can find a small $\epsilon > 0$ such that $\#U(r) < C(n, D, v) r^n$. This is a contradiction. □

Next we discuss the abelian coverings.

Let X be a compact metric space or a Riemannian manifold with $diam(X) = D$ and let $\widetilde{X}$ be its universal covering. Set $H = [\pi_1(X), \pi_1(X)]$, $, = \pi_1(X)/H$ and $\hat{X} = \widetilde{X}/H$. Then $\hat{X}$ is a covering space of X and , acts on $\hat{X}$ as an abelian group of isometries. Fix a point $p \in \hat{X}$.

Lemma 2 ([Col], [Ya3]). *There exists a subgroup Λ of , of finite index which is isomorphic to $\mathbf{Z}^{b_1(X)}$; Furthermore for all $\gamma \in \Lambda - \{1\}$, $D \leq d(\gamma(p), p)$ and for all i, $d(\gamma_i(p), p) \leq 2D$, where $\{\gamma_i\}$ is a finite system of generators of Λ.*

Finally here we are going to sketch the proof for Theorem 2.3.

Proof of Theorem 2.3. Suppose that it does not hold. Then there would exist a sequence of closed Riemannian n-manifolds M_i such that

$$K_{M_i} > \epsilon_i \to 0, \quad diam(M) = 1, \quad b_1 \equiv b_1(M_i) > 0,$$

and that no finite covers of M_i fiber over a b_1-dimensional torus. Let $\hat{M}_i \to M_i$ be the abelian coveing as above, and $p_i \in \hat{M}_i$. By Theorem A, we may assume that $(\hat{M}_i, p_i, ,\ _i)$ converges to (X, x_o, G) with respect to the pointed Gromov-Hausdorff topology, where $,\ _i = \pi_1(M_i)/[\pi_1(M_i), \pi_1(M_i)]$. By splitting theorem (Theorem B), X is isometric to $Y\times$ $\mathbf{R}^k$, where Y does not contain a line.

Lemma 2 shows that we may assume by rescaling metrics $(\hat{M}_i, p_i, \Lambda_i)$ converges to $(\mathbf{R}^k, 0, \Lambda)$, where Λ_i is a subgroup of $,\ _i$ with finite index and Λ is a free abelina group of rank b_1.

Using the notion of pseudogroup, one can show that a finite cover M_i^* of $M_i$converges to a flat k-torus and fibration theorem due to Yamaguchi implies an fibering of M_i^* over the k-torus. For more details, see [Ya2] or [Ya3].

This completes the proof of Theorem 2.3. □

The proofs for other Theorems are quite long and so we omit these. For proofs, see the original papers.

§4. splitting theorem of Ricci version

To extend the well-known resluts on manifolds of non-negative Ricci curvture to manifolds of almost non-negative Ricci curvture, a major problem is the *splitting phenomina* which is conjectured by Fukaya-Yamaguch and proved by Cheeger and Colding.

Theorem 4.1 ([C-C1])**.** *Let (X, d) be the pointed Gromov-Hausdorff limit of a sequence (M_i, g_i) of complete Riemannian n-manifolds with $Ric(M_i) \geq -\epsilon_i \to 0$. Then the splitting theorem holds for X, i.e., if X contains a line, then X splits into* $\mathbf{R}$ $\times X'$ *isometrically.*

Using this theorem, many people could prove several results similar to the case of non-negative Ricci curvature manifolds.

First Cheeger and Colding extended Theorem 2.4 as follows which is originally conjectured by Gromov.

Theorem 4.2 ([C-C1])**.** *There exists a positive number $\epsilon = \epsilon(n)$, depending only on the dimension n such that if*

$$Ric(M) \cdot diam(M)^2 > -\epsilon,$$

then $\pi_1(M)$ is almost nilpotent.

The author extends a Wei's result (Theorem 2.5)

Theorem 4.3 ([Yun])**.** *Given n and $D, v > 0$, there exists a positive number $\epsilon = \epsilon(n, D, v)$, depending only on n, D and v, such that if*

$$Ric(M) > -\epsilon, \quad dima(M) \leq D, \quad vol(M) > v,$$

then $\pi_1(M)$ is almost abelian.

On the other hand, Colding proved the following

Theorem 4.4 ([Col]). *There exists a positive number $\epsilon = \epsilon(n)$, depending ony on the dimension n such that if (M, g) is a closed Riemannian n-manifold with $Ric(M) \cdot diam(M)^2 > -\epsilon$ and $b_1(M) = n$, then M is homeomorphic to a torus if $n \neq 3$ and homotopically equivalent to T^n if $n = 3$.*

Finally, we close this section with a statement of a fibration theorem due to T. Yamaguchi. In [Ya1], Yamaguchi proved a structure theorem for manifolds of almost non-negative Ricci curvature which is an extension of Theorem 2.3 in some sense.

Theorem 4.5 ([Ya1]). *Given n and $D, \Lambda > 0$, there exists a positive number $\epsilon = \epsilon(n, D, \Lambda)$ such that if (M, g) is a closed Riemannian n-manifold satisfying*

$$|K_M| \leq \Lambda, \quad diam(M) < D, \quad Ric(M) > -\epsilon, \tag{*}$$

then M is a fiber bundle over a $b_1(M)$-torus. In particular, if $b_1(M) = n - 1$, then M is diffeomorphic to an infranilmanifold, and if $b_1(M) = n$, then M is diffoemorphic to an n-torus.

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