Trends in Mathematics
Information Center for Mathematical Sciences
Volume 1, Number 1, September 1998, Pages 000–000

ENUMERATION OF GRAPH COVERINGS AND RELATED SURFACE TOPOLOGY

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ABSTRACT. In this note, we aim to give several kinds of enumeration results about graph coverings and some of their applications to a surface topology.

1. DEFINITIONS AND NOTATIONS

G: a connected finite simple graph with vertex set $V(G)$ and edge set $E(G)$. **Neighborhood** $N(v)$ of v : the set of vertices adjacent to $v \in V(G)$. A map $p : \widetilde{G} \to G$ is a **covering** of G if $p : V(\widetilde{G}) \to V(G)$ is surjective and $p|_{N(\tilde{v})} : N(\tilde{v}) \to N(v)$ is bijective for $\forall v \in V(G)$ and $\forall \tilde{v} \in p^{-1}(v)$. It is an n**-fold covering** if p is n-to-one. A covering $p : \widetilde{G} \to G$ is **regular** (or, $\mathcal{A}$**-covering**) if there is a subgroup $\mathcal{A}$ of the automorphism group $\mathrm{Aut}\,(\widetilde{G})$ of $\widetilde{G}$ acting freely on $\widetilde{G}$ so that the graph G is isomorphic to the quotient graph $\widetilde{G}/\mathcal{A}$, say by h, and the quotient map $\widetilde{G} \to \widetilde{G}/\mathcal{A}$ is the composition $h \circ p$ of p and h.

Betti number $\beta(G)$ of G: the number of independent cycles in G. $D(G)$: the set of all directed edges risen from the edges in $E(G)$. $\mathrm{Iso}\,(G; n)$ (resp. $\mathrm{Iso}^R(G; n)$): the number of isomorphism classes of (resp. **regular**) n-fold coverings of G. $\mathrm{Isoc}\,(G; n)$ (resp. $\mathrm{Isoc}^R(G; n)$): the number of isomorphism classes of **connected** (resp. **regular**) n-fold coverings of G.

$\mathrm{Iso}\,(G; \mathcal{A})$ (resp. $\mathrm{Isoc}\,(G; \mathcal{A})$): the number of isomorphism classes of (resp. **connected**) $\mathcal{A}$-coverings of G.

S_n, $\mathbb{Z}_n$ and $\mathbb{D}_n$: the symmetric group on n letters, the cycle group of order n and the dihedral group of order $2n$, respectively.

Permutation voltage assignment of G: a function ϕ from $D(G)$ into S_n with the property that $\phi(vu) = \phi(uv)^{-1}$ for each $uv \in D(G)$.

Ordinary $\mathcal{A}$**-voltage assignment** of G: a function ϕ from $D(G)$ into a group $\mathcal{A}$ with the property that $\phi(vu) = \phi(uv)^{-1}$ for each $uv \in D(G)$.

Permutation derived graph G^ϕ : $V(G^\phi) = V(G) \times \{1, 2, \cdots, n\}$, and (u, i) and (v, j) are adjacent iff $e = uv \in D(G)$ and $j = \phi(e)(i)$. The first coordinate projection $p^\phi : G^\phi \to G$ is an n-fold covering.

Ordinary derived graph $G \times_\phi \mathcal{A}$: $V(G \times_\phi \mathcal{A}) = V(G) \times \mathcal{A}$, and (u, g) and (v, h) are adjacent iff $e = uv \in D(G)$ and $h = \phi(e)g$. The first coordinate projection $p_\phi : G \times_\phi \mathcal{A} \to G$ is an $\mathcal{A}$-covering.

1991 *Mathematics Subject Classification.* 57M12, 57M15, 05C30.
Key words and phrases. graph, enumeration, surface (branched) covering, Alexander Theorem.

Two coverings $p_i : \widetilde{G}_i \to G$, $i = 1, 2$, are said to be **isomorphic** if there exists a graph isomorphism $\Phi : \widetilde{G}_1 \to \widetilde{G}_2$ such that $p_2 \circ \Phi = p_1$. Such a Φ is called a **covering isomorphism**.

2. Enumeration formulas

[1]. ([?]) Any n-fold covering of G is isomorphic to a permutation derived graph G^{ϕ} for some $\phi \in C^1_T(G; n)$, and any regular n-fold covering of G is isomorphic to an ordinary derived graph $G \times_{\phi} \mathcal{A}$ for some group $\mathcal{A}$ with $|\mathcal{A}| = n$ and $\phi \in C^1_T(G; \mathcal{A})$, where $C^1_T(G; n)$ (resp. $C^1_T(G; \mathcal{A})$) denotes the set of all permutation (resp. ordinary $\mathcal{A}$-) voltage assignments ϕ of G which are trivial on the edges of a fixed spanning tree T of G.
[2]. ([?]) Two coverings $p^{\phi} : G^{\phi} \to G$ and $p^{\psi} : G^{\psi} \to G$ are isomorphic if and only if there exists a function $f : V(G) \to S_n$ such that $\psi(uv) = f(v)\phi(uv)f(u)^{-1}$ for each $uv \in D(G)$. Moreover, if $\phi, \psi \in C^1_T(G; n)$, then it is equivalent to say that there exists a permutation $\sigma \in S_n$ such that $\psi(uv) = \sigma\phi(uv)\sigma^{-1}$ for each $uv \in D(G) - D(T)$.
[3]. ([?]) For a partition $\mathfrak{p}$ of n, let $j_k(\mathfrak{p})$ denote the multiplicity of k in the partition $\mathfrak{p}$, so that $j_1(\mathfrak{p}) + 2j_2(\mathfrak{p}) + \cdots + nj_n(\mathfrak{p}) = n$. For convenience, let $\mathfrak{P}(n)$ denote the set of all partitions of the natural number n. Note that for any n-fold covering $p : \tilde{G} \to G$, the fold numbers of $p|_{\tilde{G}_i} : \tilde{G}_i \to G$ for the components G_i of $\tilde{G}$ gives rise to a partition $\mathfrak{p}$ of n, called the component type of the covering p. For a partition $\mathfrak{p}$ of n, let $\mathrm{Iso}\,(G; \mathfrak{p})$ denote the number of isomorphism classes of n-fold coverings of G having the component type $\mathfrak{p}$. Clearly, $\mathrm{Iso}\,(G; \mathfrak{p}) = \mathrm{Isoc}\,(G; n)$ if $j_n(\mathfrak{p}) = 1$, and

$$\mathrm{Iso}\,(G; n) = \sum_{\mathfrak{p} \in \mathfrak{P}(n)} \mathrm{Iso}\,(G; \mathfrak{p}).$$

[4]. ([?]) For any natural number n, $\mathrm{Iso}^R(G; n) = \sum_{d|n} \mathrm{Isoc}^R(G; d)$.
[5]. ([?]) For any natural number n, $\mathrm{Isoc}^R(G; n) = \sum_{\mathcal{A}} \mathrm{Isoc}\,(G; \mathcal{A})$, where $\mathcal{A}$ runs over all representatives of isomorphism classes of groups of order n.
[6]. ([?]) For any two voltage assignments $\phi \in C^1_T(G; \mathcal{A})$ and $\psi \in C^1_T(G; \mathcal{B})$, if their derived (regular) coverings $p_{\phi} : G \times_{\phi} \mathcal{A} \to G$ and $p_{\psi} : G \times_{\psi} \mathcal{B} \to G$ are connected, then they are equivalent if and only if there exists a group isomorphism $\sigma : \mathcal{A} \to \mathcal{B}$ such that $\psi(uv) = \sigma(\phi(uv))$ for all $uv \in D(G) - D(T)$. From this characterization, we can show that for any finite group $\mathcal{A}$,

$$\mathrm{Isoc}\,(G; \mathcal{A}) = \frac{|\mathfrak{G}(\mathcal{A}; \beta(G))|}{|\mathrm{Aut}\,(\mathcal{A})|},$$

where $\mathfrak{G}(\mathcal{A}; n) = \{\, (g_1, g_2, \cdots, g_n) \mid \{g_1, g_2, \cdots, g_n\} \text{ generates } \mathcal{A}\}$.
[7]. ([?]) For any finite group $\mathcal{A}$, $\mathrm{Iso}\,(G; \mathcal{A}) = \sum_{\mathcal{S}} \mathrm{Isoc}\,(G; \mathcal{S})$, where $\mathcal{S}$ runs over all representatives of isomorphism classes of subgroups of $\mathcal{A}$.
[8]. ([?]) For any finite groups $\mathcal{A}$, $\mathcal{B}$ with $(|\mathcal{A}|, |\mathcal{B}|) = 1$,

$$\mathrm{Iso}\,(G; \mathcal{A} \oplus \mathcal{B}) = \mathrm{Iso}\,(G; \mathcal{A})\ \mathrm{Iso}\,(G; \mathcal{B}) \ \text{ and } \ \mathrm{Isoc}\,(G; \mathcal{A} \oplus \mathcal{B}) = \mathrm{Isoc}\,(G; \mathcal{A})\ \mathrm{Isoc}\,(G; \mathcal{B}).$$

3. Numerical results

[1]. ([?], [?]) The number of isomorphism classes of n-fold coverings of G is

$$\mathrm{Iso}\,(G;n)=\sum_{\ell_1+2\ell_2+\cdots+n\ell_n=n}(\ell_1!\,2^{\ell_2}\ell_2!\,\cdots n^{\ell_n}\ell_n!)^{\beta(G)-1}.$$

[2]. ([?]) The number of isomorphism classes of connected n-fold, $n\geq 2$, coverings of G is

$$\begin{aligned}\mathrm{Isoc}\,(G;n)&\\ =&\sum_{\ell_1+2\ell_2\cdots+(n-1)\ell_{n-1}=n-1}\Big((\ell_1+1)^{\beta(G)-1}-1\Big)\left(\ell_1!\,2^{\ell_2}\ell_2!\,\cdots(n-1)^{\ell_{n-1}}\ell_{n-1}!\right)^{\beta(G)-1}\\ &+\sum_{2\ell_2+3\ell_3+\cdots+n\ell_n=n}\left(2^{\ell_2}\ell_2!\,3^{\ell_3}\ell_3!\,\cdots n^{\ell_n}\ell_n!\right)^{\beta(G)-1}\\ &-\sum_{\substack{\mathfrak{p}\,\in\,\mathfrak{P}(n)\\ j_1(\mathfrak{p})\,=\,0;\,j_n(\mathfrak{p})\,\neq\,1}}\ \prod_{j_k(\mathfrak{p})\neq 0}\left(\frac{1}{j_k(\mathfrak{p})!}\prod_{\ell=0}^{j_k(\mathfrak{p})-1}(\mathrm{Isoc}\,(G;k)+\ell)\right),\end{aligned}$$

where the summation over the empty index set is defined to be 0. For example,

$$\mathrm{Isoc}\,(G;2)=2^{\beta(G)}-1,$$
$$\mathrm{Isoc}\,(G;3)=6^{\beta(G)-1}+3^{\beta(G)-1}-2^{\beta(G)-1},$$
$$\mathrm{Isoc}\,(G;4)=24^{\beta(G)-1}+8^{\beta(G)-1}-6^{\beta(G)-1}.$$

[3]. ([?], [?]) The number of isomorphism classes of $\mathbb{Z}_n$-coverings of G is

$$\mathrm{Iso}\,(G;\mathbb{Z}_n)=\begin{cases}1 & \text{if } \beta(G)=0,\\ \displaystyle\prod_{i=1}^{\ell}(s_i+1) & \text{if } \beta(G)=1,\\ \displaystyle\prod_{i=1}^{\ell}\left(1+\frac{(p_i^{\beta(G)}-1)(p_i^{s_i(\beta(G)-1)}-1)}{(p_i-1)(p_i^{\beta(G)-1}-1)}\right) & \text{if } \beta(G)\geq 2\end{cases}$$

for $n=p_1^{s_1}p_2^{s_2}\cdots p_\ell^{s_\ell}$ (a prime factorization). Moreover, [?], [?])

$$\mathrm{Iso}\,(G;m\mathbb{Z}_p)=1+\sum_{i=1}^{m}\prod_{j=1}^{i}\frac{p^{\beta(G)-j+1}-1}{p^j-1}\quad\text{for a prime } p.$$

[4]. ([?]) The number of isomorphism classes of connected $\mathbb{Z}_n$-coverings of G is

$$\mathrm{Isoc}\,(G;\mathbb{Z}_n)=\begin{cases}0 & \text{if } \beta(G)=0,\\ \displaystyle\prod_{i=1}^{\ell}p_i^{(\beta(G)-1)(s_i-1)}\frac{p_i^{\beta(G)}-1}{p_i-1} & \text{if } \beta(G)\geq 1\end{cases}$$

for $n=p_1^{s_1}p_2^{s_2}\cdots p_\ell^{s_\ell}$. Moreover,

$$\mathrm{Isoc}\,(G;\oplus_{h=1}^{\ell}m_h\mathbb{Z}_{p^{s_h}})=p^{f(\beta(G),m_i,s_i)}\frac{\displaystyle\prod_{i=1}^{m}p^{\beta(G)-i+1}-1}{\displaystyle\prod_{j=1}^{\ell}\prod_{h=1}^{m_j}p^{m_j-h+1}-1},$$

where $m = m_1 + \cdots + m_\ell$, p is prime, $m_h\mathbb{Z}_{p^{s_h}}$ is the direct sum of m_h copies of the cyclic group $\mathbb{Z}_{p^{s_h}}$ for each natural numbers h and s, and

$$f(\beta(G), m_i, s_i) = (\beta(G) - m)\left(\sum_{i=1}^{\ell} m_i(s_i - 1)\right) + \sum_{i=1}^{\ell-1} m_i \left(\sum_{j=i+1}^{\ell} m_j(s_i - s_j - 1)\right).$$

In particular,

$$\mathrm{Isoc}\,(G; m\mathbb{Z}_p) = \prod_{i=1}^{m} \frac{p^{\beta(G)-i+1} - 1}{p^i - 1}.$$

NOTE: By using 2.5-2.8 and the computation of $\mathrm{Isoc}\,(G; \oplus_{h=1}^{\ell} m_h\mathbb{Z}_{p^{s_h}})$, we can compute the number $\mathrm{Iso}\,(G; \mathcal{A})$ for any finite abelian group $\mathcal{A}$. For example,

$$\begin{aligned}
&\mathrm{Iso}\,(G; \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p)\\
&= \mathrm{Isoc}\,(G; \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p) + \mathrm{Isoc}\,(G; \mathbb{Z}_p \oplus \mathbb{Z}_p) + \mathrm{Iso}\,(G; \mathbb{Z}_{p^2})\\
&= \frac{(p^{\beta(G)} - 1)(p^{\beta(G)-1} - 1)}{(p^2 - 1)(p - 1)}\left(p^{\beta(G)-1} + p^{\beta(G)-2} + 1\right) + \frac{p^{\beta(G)} - 1}{p - 1}\left(p^{\beta(G)-1} + 1\right) + 1.
\end{aligned}$$

[5]. ([?] [?]) The number of isomorphism classes of (connected) $\mathbb{D}_n$-coverings of G is

$$\mathrm{Iso}\,(G; \mathbb{D}_n) = \begin{cases} \mathrm{Iso}\,(G; \mathbb{Z}_n) + (2^{\beta(G)} - 1)\mathrm{Iso}\,(G - e; \mathbb{Z}_n) & \text{if } n \text{ is odd,} \\ \mathrm{Iso}\,(G; \mathbb{Z}_n) + (2^{\beta(G)} - 1)\mathrm{Iso}\,(G - e; \mathbb{Z}_n) - \frac{1}{3}(4^{\beta(G)} - 1) & \text{if } n \text{ is even,} \end{cases}$$

for $n \geq 3$ and an edge e in the cotree $G - T$. And

$$\mathrm{Isoc}\,(G; \mathbb{D}_n) = (2^{\beta(G)} - 1)\prod_{i=1}^{\ell} p_i^{(s_i-1)(\beta(G)-2)} \frac{p_i^{\beta(G)-1} - 1}{p_i - 1} \quad \text{for} \quad n = p_1^{s_1} p_2^{s_2} \cdots p_\ell^{s_\ell} \geq 3.$$

[6].([?],[?], [?]) For a prime p, the numbers of isomorphism classes of some (connected) regular coverings of G are

$$\mathrm{Iso}^R(G; p) = \frac{p^{\beta(G)} - 1}{p - 1} + 1,$$

$$\mathrm{Iso}^R(G; p^2) = \frac{(p^{\beta(G)} - 1)(p^{\beta(G)-1} - 1)}{(p^2 - 1)(p - 1)} + \frac{p^{\beta(G)} - 1}{p - 1}(p^{\beta(G)-1} + 1) + 1,$$

$$\begin{aligned}
\mathrm{Iso}^R(G; p^3) &= \frac{(p^{\beta(G)} - 1)(p^{\beta(G)-1} - 1)(p^{\beta(G)-2} - 1)}{(p^3 - 1)(p^2 - 1)(p - 1)}\\
&+ \tfrac{(p^{\beta(G)} - 1)(p^{\beta(G)-1} - 1)}{2(p^2 - 1)(p - 1)}\left(p^{\beta(G)+1} + p^{\beta(G)-1} + 4 \cdot p^{\beta(G)-2} + 2\right)\\
&+ \tfrac{p^{\beta(G)} - 1}{p - 1}\left(p^{2(\beta(G)-1)} + p^{\beta(G)-1} + 1\right) + 1,
\end{aligned}$$

$$\mathrm{Isoc}^R(G; p) = \frac{p^{\beta(G)} - 1}{p - 1},$$

$$\mathrm{Isoc}^R(G; p^2) = \frac{(p^{\beta(G)} - 1)(p^{\beta(G)-1} - 1)}{(p^2 - 1)(p - 1)} + p^{(\beta(G)-1)}\frac{p^{\beta(G)} - 1}{p - 1},$$

$$\begin{aligned}\mathrm{Isoc}^R(G;p^3) &= \frac{(p^{\beta(G)}-1)(p^{\beta(G)-1}-1)(p^{\beta(G)-2}-1)}{(p^3-1)(p^2-1)(p-1)} \\ &+ p^{\beta(G)-2}\tfrac{(p^{\beta(G)}-1)(p^{\beta(G)-1}-1)}{(p^2-1)(p-1)}(p+2) \\ &+ p^{\beta(G)-1}\tfrac{p^{\beta(G)}-1}{2(p-1)}\left(3p^{\beta(G)-1}-1\right).\end{aligned}$$

4. Applications

[1]. ([?]) Isoc $(G;n)$ *and the conjugacy classes of subgroups of a free group* It is well-known (e.g., see [?]) in topology that the fundamental group of a graph G is a free group of rank $\beta(G)$, and there exists a one-to-one correspondence between the isomorphism classes of connected n-fold coverings of G and the conjugacy classes of subgroups of index n of the fundamental group of G. Thus, by using the enumerating formula for Isoc $(G;n)$, we can compute the total number of conjugacy classes of subgroups of index n of a free group generated by $\beta(G)$ elements.

[2]. ([?]) $\sum_{\mathcal{A}} \mathrm{Isoc}^R(G;\mathcal{A})$ *and the subgroups of a free abelian group* For a connected $\mathcal{A}$-covering $p:\tilde{G}\to G$, the image $p_*(\pi_1(\tilde{G}))$ of the fundamental group of the covering graph $\tilde{G}$ is a normal subgroup of the fundamental group $\pi_1(G)$ of the base graph G, and the quotient group $\pi_1(G)/p_*(\pi_1(\tilde{G}))$ is isomorphic to $\mathcal{A}$. If $\mathcal{A}$ is abelian, then $p_*(\pi_1(\tilde{G}))$ contains the commutator subgroup $[\pi_1(G),\pi_1(G)]$ of the free group $\pi_1(G)$ generated by $\beta(G)$ elements. Since $[\pi_1(G),\pi_1(G)]$ is a normal subgroup of $\pi_1(G)$, the natural homomorphism $q:\pi_1(G)\to\pi_1(G)/[\pi_1(G),\pi_1(G)]$ induces a one-to-one correspondence between the set of all subgroups of $\pi_1(G)$ containing $[\pi_1(G),\pi_1(G)]$ and the set of all subgroups of the quotient group $\pi_1(G)/[\pi_1(G),\pi_1(G)]$. Notice that $\pi_1(G)/[\pi_1(G),\pi_1(G)]$ is the free abelian group generated by $\beta(G)$ elements. Now, from a well-known classification theorem for regular coverings of a topological space, it follows that the number $\sum_{\mathcal{A}} \mathrm{Isoc}^R(G;\mathcal{A})$, where $\mathcal{A}$ runs over all representatives of isomorphism classes of abelian groups of order n, is equal to the number of subgroups of index n of the free abelian group generated by $\beta(G)$ elements.

[3]. ([?]) Isoc $(G;m\mathbb{Z}_p)$ *and the subspaces of the n-dimensional vector space over the field $\mathbb{Z}_p$* It is well-known that the number of the m-dimensional subspaces of the n-dimensional vector space $n\mathbb{Z}_p$ over the field $\mathbb{Z}_p$ is equal to the Gaussian coefficient

$$\left[\begin{array}{c} n \\ m \end{array}\right]_p = \frac{\prod_{i=n-m+1}^{n}(p^i-1)}{\prod_{i=1}^{m}(p^i-1)}.$$

Hence, we can say that the number Isoc $(G;m\mathbb{Z}_p)$ of the isomorphism classes of connected $m\mathbb{Z}_p$-coverings of a graph G is equal to the number of the m-dimensional subspaces of the $\beta(G)$-dimensional vector space $\beta(G)\mathbb{Z}_p$.

[4]. *Distributions of branched surface coverings* A **surface** $\mathbb{S}$ is a compact connected 2-manifold without boundary. By the classification of surfaces, a surface $\mathbb{S}$

is homeomorphic to one of the following:

$$\mathbb{S}_k = \begin{cases} \text{the orientable surface with } k \text{ handles} & \text{if } k > 0, \\ \text{the sphere } \mathbb{S}^2 & \text{if } k = 0, \\ \text{the nonorientable surface with } -k \text{ crosscaps} & \text{if } k < 0. \end{cases}$$

A continuous surjective map $p : \tilde{\mathbb{S}} \to \mathbb{S}$ is a **branched covering** if $p|_{\tilde{\mathbb{S}}-F} : \tilde{\mathbb{S}} - F \to \mathbb{S} - p(F)$ is a covering for some finite subset F of $\tilde{\mathbb{S}}$. The **branch set** B of a branched covering $p : \tilde{\mathbb{S}} \to \mathbb{S}$ is the collection of points $x \in \mathbb{S}$ which have the property that x has no neighborhood N_x such that each component of $p^{-1}(N_x)$ is mapped homeomorphically onto N_x by p. A branched covering $p : \tilde{\mathbb{S}} \to \mathbb{S}$ is regular (or $\mathcal{A}$-covering) if $p|_{\tilde{\mathbb{S}}-p^{-1}(B)} : \tilde{\mathbb{S}} - p^{-1}(B) \to \mathbb{S} - B$ is a regular covering (with the covering transformation group $\mathcal{A}$). Two branched coverings $p_i : \tilde{\mathbb{S}}_i \to \mathbb{S}$ $(i = 1, 2)$ are **equivalent** if there exists a homeomorphism $\tilde{h} : \tilde{\mathbb{S}}_1 \to \tilde{\mathbb{S}}_2$ such that $p_2 \circ \tilde{h} = p_1$. A well-known theorem of Alexander ([**?**]) says that every orientable surface is a branched covering of the sphere $\mathbb{S}^2$, and every nonorientable surface is a branched covering of the projective plane. In the study of surface branched coverings, we can ask naturally as a generalization of Alexander's theorem: *In how many different ways can a given surface be a branched covering of another given surface?* To give a systematic answer of this question, we define two polynomials, called **branched covering distribution polynomials** in the following. In some sense, this gives a classification of the pseudo-free group actions on a surface when a data for a quotient surface is given. For each $i \in \mathbb{Z}$, let $a_i(\mathbb{S}, B, \mathcal{A})$ denote the number of equivalence classes of branched $\mathcal{A}$-coverings $p : \mathbb{S}_i \to \mathbb{S}$ with branch set B, and let

$$R_{(\mathbb{S},B,\mathcal{A})}(x) = \sum_{i=-\infty}^{\infty} a_i(\mathbb{S}, B, \mathcal{A}) x^i.$$

This polynomial can have at most finitely many nonzero terms by the Riemann-Hurwitz equation: $\chi(\tilde{\mathbb{S}}) = |\mathcal{A}|\chi(\mathbb{S}) - \sum_{b \in B} \operatorname{def}(b)$, where $\operatorname{def}(b) = |\mathcal{A}| - |p^{-1}(b)|$ and χ denotes the Euler characteristic.

Theorem: When $\mathcal{A} = \mathbb{Z}_p$, $p =$ a prime ([**?**]): 1. Let $\mathbb{S}_k,\ k \geq 0$, be an orientable surface, B a finite subset of $\mathbb{S}_k$ and p any prime. Then

$$a_i(\mathbb{S}_k, B, \mathbb{Z}_p)$$

$$= \begin{cases} \dfrac{p^{2k} - 1}{p - 1} & \text{if } i = 1 + p(k-1),\ |B| = 0, \\ p^{2k-1}\left((p-1)^{|B|-1} + (-1)^{|B|}\right) & \text{if } i = pk + \dfrac{p-1}{2}(|B| - 2), |B| \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

2. Let $\mathbb{S}_k,\ k < 0$, be a nonorientable surface and B a finite subset of $\mathbb{S}_k$. Then

$$a_i(\mathbb{S}_k, B, \mathbb{Z}_2) = \begin{cases} 1 & \text{if } i = -k-1,\ |B| = 0, \\ 2^{-k} - 2 & \text{if } i = 2(k+1),\ k \neq -1,\ |B| = 0, \\ 2^{-k} & \text{if } i = 2(k+1) - |B|,\ |B| \neq 0,\ |B| = \text{even}, \\ 0 & \text{otherwise.} \end{cases}$$

3. Let $\mathbb{S}_k,\ k < 0$, be a nonorientable surface, B a finite subset of $\mathbb{S}_k$ and p an odd prime. Then

$$a_i(\mathbb{S}_k, B, \mathbb{Z}_p) = \begin{cases} \dfrac{p^{-k-1} - 1}{p-1} & \text{if } i = p(k+2) - 2,\ |B| = 0, \\ p^{-k-1}(p-1)^{|B|-1} & \text{if } i = p(k+2) - |B|(p-1) - 2,\ |B| \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

4. When $\mathcal{A} = \mathbb{D}_p$, $p =$ a prime, see ([**?**]).

Some interesting topological results: A group $\mathcal{A}$ action on a surface $\mathbb{S}$ is **pseudofree** if the number of fixed points of the action is finite, *i.e.,* the cardinality of the set $\{x \in \mathbb{S} \mid gx = x \text{ for some } g \neq id \text{ in } \mathcal{A}\}$ is finite. A group action on a surface is **spherical** if the quotient surface of the action is homeomorphic to the sphere.

1. For any $k \geq 0$, there are exactly $4^k - 1$ nonequivalent connected unbranched double coverings of $\mathbb{S}_k$, and all of their covering surfaces are $\mathbb{S}_{2k-1}$.
2. For any surface $\mathbb{S}$, there does not exist a connected branched double covering of $\mathbb{S}$ with odd number of branch points.
3. For any $k \geq 0$ and even number $2b$, $b \geq 1$, there are exactly 4^k nonequivalent connected branched double coverings of $\mathbb{S}_k$ having given $2b$ branch points, and all of their covering surfaces are $\mathbb{S}_{2k+b-1}$.
4. There exists a unique connected unbranched double covering of the projective plain $\mathbb{S}_{-1}$ up to equivalence, and its covering surface is the sphere. For any $k \leq -2$, there exist $2^{-k} - 1$ connected unbranched double coverings of $\mathbb{S}_k$ up to equivalence, and one of their covering surfaces is the orientable surface $\mathbb{S}_{-k-1}$ and all others are the nonorientable surface $\mathbb{S}_{2(k+1)}$.
5. For any $k \leq -1$ and even number $2b$, $b \geq 1$, there are exactly 2^{-k} nonequivalent connected branched double coverings of $\mathbb{S}_k$ having given $2b$ branch points, and all of their covering surfaces are the nonorientable surface $\mathbb{S}_{2(k-b+1)}$.
6. Every orientable surface is a branched double covering of the sphere $\mathbb{S}^2$. Every nonorientable surface is a branched branched double or triple covering of the projective plane $\mathbb{S}_{-1}$ (This is Alexander's theorem).
7. $\sum_{m=0}^{\infty} \bar{a}_i(\mathbb{S}_0, m, \mathbb{Z}_p)$ stands for the number of weak equivalence classes of pseudofree spherical $\mathbb{Z}_p$-actions on the surface $\mathbb{S}_i$. And this number can be computed. In particular, any pseudofree spherical $\mathbb{Z}_p$-action on the sphere is weakly equivalent to the group action generated by the rotation through $\frac{2\pi}{p}$ about the z-axis.
8. Any two free $\mathbb{Z}_p$-actions on a surface $\mathbb{S}$ with orientable quotient surface $\mathbb{S}/\mathbb{Z}_p$ are weakly equivalent. The number of weak equivalence classes of pseudofree $\mathbb{Z}_2$-actions on an orientable surface $\mathbb{S}_k$ is $\lfloor \frac{k+1}{2} \rfloor + 2$. Moreover, the quotient surface $\mathbb{S}_k/\mathbb{Z}_2$ is homeomorphic to one of the following $\lfloor \frac{k+1}{2} \rfloor + 2$ surfaces: $\mathbb{S}_0, \mathbb{S}_1, \cdots, \mathbb{S}_{\lfloor \frac{k+1}{2} \rfloor}$ and

a nonorientable surface $\mathbb{S}_{-k-1}$.

9. Let p be prime ≥ 2. Then the dihedral group $\mathbb{D}_p$ can act freely on the surface $\mathbb{S}_k$ if and only if either $k \geq 1$ and $k-1 \equiv 0 \pmod p$ or $k \leq -3$ and $k+2 \equiv 0 \pmod{2p}$. Moreover, i. if $k \geq 1$, $k-1 \equiv 0 \pmod p$ and $k-1 \not\equiv 0 \pmod{2p}$, then $\mathbb{S}_k/\mathbb{D}_p$ is the nonorientable surface $\mathbb{S}_{\frac{1-k}{p}-2}$; ii. if $k \geq 2$ and $k-1 \equiv 0 \pmod{2p}$, then $\mathbb{S}_k/\mathbb{D}_p$ is either the orientable surface $\mathbb{S}_{\frac{k-1}{2p}+1}$ or the nonorientable surface $\mathbb{S}_{\frac{1-k}{p}-2}$; iii. if $k \leq -3$ and $k+2 \equiv 0 \pmod{2p}$, then $\mathbb{S}_k/\mathbb{D}_p$ is the nonorientable surface $\mathbb{S}_{\frac{k+2}{2p}-2}$.

10. For any prime $p \geq 2$, a surface $\mathbb{S}_k$ has a spherical pseudofree $\mathbb{D}_p$-action if and only if $k = (p-1)m + n$, where $m, n \geq 0$ and $m+1 \geq n$. Moreover, for such a $k = (p-1)m + n$, the number of branch points of the $\mathbb{D}_p$-covering $p : \mathbb{S}_k \to \mathbb{S}_k/\mathbb{D}_p = \mathbb{S}_0$ is $m+n+3$.

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