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HOMOLOGY OF THE TRIPLE LOOP SPACES OF THE SIMPLE COMPACT LIE GROUPS

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The homology of the triple loop space of a Lie group G is reflected in the global nature of moduli space of G instantons. So it is reasonable to study the homology of the triple loop spaces of Lie groups.

1. Introduction

Let G be a compact, connected simple Lie group and $\pi{:}P \longrightarrow S^4$ be a principal G-bundle over S^4. Since $\pi_4(BG) = \pi_3(G) = Z$, we can classify the principle bundle P_k over S^4 by the map $S^4 \longrightarrow BG$ of degree k. The orbit spaces of all connections on P_k up to based gauge equivalence is homotopy equivalent to the triple loop space of G [1]. That is, $\mathcal{C}_k = \mathcal{A}_k/\mathcal{G}^b(P_k) \simeq \Omega^3_k G$ where $\mathcal{A}_k$ is the space of the all connections in P_k and $\mathcal{G}^b(P_k)$ is the based gauge group which consists of all base point preserving automorphisms on P_k.

Let $\mathcal{M}_k$ be the space of based gauge equivalence classes of all connections in P_k satisfying the Yang -Mills self-duality equations, which we call the moduli space of G instantons. Then there is a natural inclusion map $i : \mathcal{M}_k \to \mathcal{C}_k \simeq \Omega^3_k G$ and the inclusion map $i : \mathcal{M}_\infty \to \mathcal{C}_\infty$ induces a homotopy equivalence [11] where $\mathcal{M}_\infty$ and $\mathcal{C}_\infty$ are the direct limits under the inclusions. Especially $\mathcal{M} = \coprod_{k>0} \mathcal{M}_k$ has additional structures [3], that is, it behaves like a four fold loop space enough to define the homology operations Q_1, Q_2, Q_3 up to homotopy.

$\Omega^3_k G$ is infinite dimensional and each $\Omega^3_k G$ is homotopy equivalent to $\Omega^3_0 G$ for any component k. However $\mathcal{M}_k$ is finite dimensional and the dimension of $\mathcal{M}_k$ increases as k increases. Moreover whenever k increases, the homology of triple loop space of G is contained in the homology of the G insatntons space. So it is reasonable to study the homology of triple loop space of G to get the information about the homology of instantons space and its gauge group [2],[3],[4].

2. Preliminaries

Let $E(x)$ be the exterior algebra on x, $P(x)$ be the polynomial algebra on x and $\Gamma(x)$ be the divided power algebra on x which is free over $\gamma_i(x)$ with the product $\gamma_i(x)\gamma_j = \binom{i+j}{j}\gamma_{i+j}$. Throughout this paper, the subscript of an element always means the degree of an element, for example, the degree of a_i is i. For an $(n+1)$-fold loop space, there are homology operations,

$$Q_{i(p-1)} : H_q(\Omega^{n+1}X; \mathbb{F}_p) \longrightarrow H_{pq+i(p-1)}(\Omega^{n+1}X; \mathbb{F}_p)$$

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defined for $0 \leq i \leq n$ when $p = 2$ and for $0 \leq i \leq n$, $i \equiv q$ for mod 2 when p is an odd prime which is natural for an $(n+1)$-fold loop space. Let Q_i^a be the iterated operation $Q_i \dots Q_i($ a times $)$ and β be the mod p Bockstein operation. We refer the book [9] for the condensed treatment of these homology operations.

The space of connections is a linear space. So it is not interesting from the topological view point. But Atiyah and Jone made the seminal observation that the space of connections factoring out by the gauge group, $\mathcal{C}_k$, is a homotopy equivalence to $\Omega_k^3 G$. They also consider the inclusion map $i_k : \mathcal{M}_k \to \mathcal{C}_k$ by forgetting the self dual condition and prove that

Theorem 1. [1] *For $G = SU(2)$, the inclusion i_k induces a map in homology $(i_k)_* : H_q(\mathcal{M}_k) \to H_q(\Omega_k^3 SU(2))$ which , for $k \gg q$, is a projection onto a direct summand.*

Moreover they made the following conjecture

Theorem 2[The Atiyah–Jones Conjecture]. [1] *For all $k > 0$, the natural inclusion $i_k : \mathcal{M}_k \to \Omega_k^3 SU(2)$ is a homotopy equivalence through dimension $q = q(k) = [k/2] - 2$.*

At last Boyer, Hurtubise, Mann, and Milgram solved the conjecture [2] and Tian extended the results for $SU(n)$ [13].

The theory of iterated spaces was developed by many leading topologists including F. Cohen, I. James, P. May, J. Milgram.

As a Hopf algebra the homology of the iterated loop space of the sphere is given in [9]

Theorem 3. *$H_*(\Omega^n S^{n+r}; \mathbb{F}_2)$ is a free commutative primitively generated Hopf algebra with generators $Q_I x_r$ running over all admissible I with $\lambda(I) \leq n-1$.*

For odd prime p and $n+r$ odd, $H_*(\Omega^n S^{n+r}; \mathbb{F}_p)$ is a free commutative primitively generated Hopf algebra with generators $Q_I x_r$ running over all admissible I with $\lambda(I) \leq n-1$ and $\lambda(I) \equiv r$ mod 2.

2. Homology of the triple loop space of all simple compact Lie groups

Theorem 1. [4] *As an algebra,*

$$\begin{aligned}
&H_*(\Omega_0^3 SU(n+1); \mathbb{F}_2) = \\
&\mathbb{F}_2[Q_2^a(Q_2[1] * [-2]) : a \geq 0] \\
&\otimes \mathbb{F}_2[Q_2^a(u_{2i-2}) : a \geq 0, 1 < i \leq [\tfrac{n}{2}], i \not\equiv 0 mod 2] \\
&\otimes \mathbb{F}_2[Q_1^a Q_2^b(u_{2i-1}) : a, b \geq 0, [\tfrac{n}{2}] < i \leq n, i \not\equiv 0 mod 2] \\
&\otimes \mathbb{F}_2[Q_1^a Q_3^b(v_{4i-2}) : a, b \geq 0, [\tfrac{n}{2}] < i \leq n, i \equiv 0 mod 2].
\end{aligned}$$

$$\begin{aligned}
&H_*(\Omega_0^3 SU(n+1); \mathbb{F}_p) = \\
&\mathbb{F}_p[Q_{2(p-1)}^a(Q_{2(p-1)}[1] * [-p]) : a \geq 0] \\
&\otimes \mathbb{F}_p[Q_{2(p-1)}^a(u_{2i-2}) : a \geq 0, 1 < i \leq n, i \not\equiv 0 mod p] \\
&\otimes E[Q_{p-1}^a \beta Q_{2(p-1)}^b u_{2i-2} : a \geq 0, b > 0, [\tfrac{n}{p}] < i \leq n, i \not\equiv 0 mod p] \\
&\otimes \mathbb{F}_p[\beta Q_{p-1}^a \beta Q_{2(p-1)}^b u_{2i-2} : a, b > 0, [\tfrac{n}{p}] < i \leq n, i \not\equiv 0 mod p] \\
&\otimes E(Q_{p-1}^a Q_{3(p-1)}^b v_{2pi-3} : a, b \geq 0, [\tfrac{n}{p}] < i \leq n, i \equiv 0 mod p] \\
&\otimes \mathbb{F}_p[\beta Q_{p-1}^a Q_{3(p-1)}^b v_{2pi-3} : a > 0, b \geq 0, [\tfrac{n}{p}] < i \leq n, i \equiv 0 mod p],
\end{aligned}$$

where $[1] \in H_*(\Omega^3_1 SU(n+1);\mathbb{F}_p)$ is the image of the generator in $\tilde{H}_0(S^0)$ for the map: $S^0 \to \Omega^3 SU(n+1)$.

For odd prime p, we have the following Harris decomposition[10]

$$\begin{aligned} SU(2n) &\simeq_{(p)} (SU(2n)/Sp(n)) \times Sp(n), \\ Sp(n) &\simeq_{(p)} SO(2n+1) \end{aligned}$$

From the above decomposition we can get the followings.

Corollary 2. *As an algebra, odd prime p*

$$\begin{aligned}
&H_*(\Omega^3_0 Sp(n+1);\mathbb{F}_p) = \\
&\mathbb{F}_p[Q^a_{2(p-1)}(Q_{2(p-1)}[1] * [-p]) : a \geq 0] \\
&\otimes \mathbb{F}_p[Q^a_{2(p-1)}(u_{2i-2}) : a \geq 0, 1 < i \leq 2n+1, i \not\equiv 0 mod p, i : odd] \\
&\otimes E[Q^a_{p-1}\beta Q^b_{2(p-1)} u_{2i-2} : a \geq 0, b > 0, [\tfrac{2n+1}{p}] < i \leq 2n+1, i \not\equiv 0 mod p, i : odd] \\
&\otimes \mathbb{F}_p[\beta Q^a_{p-1}\beta Q^b_{2(p-1)} u_{2i-2} : a, b > 0, [\tfrac{2n+1}{p}] < i \leq 2n+1, i \not\equiv 0 mod p, i : odd] \\
&\otimes E(Q^a_{p-1} Q^b_{3(p-1)} v_{2pi-3} : a, b \geq 0, [\tfrac{2n+1}{p}] < i \leq 2n+1, i \equiv 0 mod p, i : odd] \\
&\otimes \mathbb{F}_p[\beta Q^a_{p-1} Q^b_{3(p-1)} v_{2pi-3} : a > 0, b \geq 0, [\tfrac{2n+1}{p}] < i \leq 2n+1, i \equiv 0 mod p, i : odd],
\end{aligned}$$

Corollary 3. *As an algebra, odd prime p*

$$\begin{aligned}
&H_*(\Omega^3_0 SO(2n+1);\mathbb{F}_p) = \\
&\mathbb{F}_p[Q^a_{2(p-1)}(Q_{2(p-1)}[1] * [-p]) : a \geq 0] \\
&\otimes \mathbb{F}_p[Q^a_{2(p-1)}(u_{2i-2}) : a \geq 0, 1 < i \leq 2n-1, i \not\equiv 0 mod p, i : odd] \\
&\otimes E[Q^a_{p-1}\beta Q^b_{2(p-1)} u_{2i-2} : a \geq 0, b > 0, [\tfrac{2n-1}{p}] < i \leq 2n-1, i \not\equiv 0 mod p, i : odd] \\
&\otimes \mathbb{F}_p[\beta Q^a_{p-1}\beta Q^b_{2(p-1)} u_{2i-2} : a, b > 0, [\tfrac{2n-1}{p}] < i \leq 2n-1, i \not\equiv 0 mod p, i : odd] \\
&\otimes E(Q^a_{p-1} Q^b_{3(p-1)} v_{2pi-3} : a, b \geq 0, [\tfrac{2n-1}{p}] < i \leq 2n-1, i \equiv 0 mod p, i : odd] \\
&\otimes \mathbb{F}_p[\beta Q^a_{p-1} Q^b_{3(p-1)} v_{2pi-3} : a > 0, b \geq 0, [\tfrac{2n-1}{p}] < i \leq 2n-1, i \equiv 0 mod p, i : odd],
\end{aligned}$$

$$\begin{aligned}
&H_*(\Omega^3_0 SO(2n+2);\mathbb{F}_p) = \\
&\mathbb{F}_p[Q^a_{2(p-1)}(Q_{2(p-1)}[1] * [-p]) : a \geq 0] \\
&\otimes \mathbb{F}_p[Q^a_{2(p-1)}(u_{2i-2}) : a \geq 0, 1 < i \leq 2n-1, i \not\equiv 0 mod p, i : odd] \\
&\otimes E[Q^a_{p-1}\beta Q^b_{2(p-1)} u_{2i-2} : a \geq 0, b > 0, [\tfrac{2n-1}{p}] < i \leq 2n-1, i \not\equiv 0 mod p, i : odd] \\
&\otimes \mathbb{F}_p[\beta Q^a_{p-1}\beta Q^b_{2(p-1)} u_{2i-2} : a, b > 0, [\tfrac{2n-1}{p}] < i \leq 2n-1, i \not\equiv 0 mod p, i : odd] \\
&\otimes E(Q^a_{p-1} Q^b_{3(p-1)} v_{2pi-3} : a, b \geq 0, [\tfrac{2n-1}{p}] < i \leq 2n-1, i \equiv 0 mod p, i : odd] \\
&\otimes \mathbb{F}_p[\beta Q^a_{p-1} Q^b_{3(p-1)} v_{2pi-3} : a > 0, b \geq 0, [\tfrac{2n-1}{p}] < i \leq 2n-1, i \equiv 0 mod p, i : odd], \\
&\otimes \mathbb{F}_p[Q^a_{2(p-1)}(u_{2n-2}) : a \geq 0] \\
&\otimes E[Q^a_{p-1}\beta Q^b_{2(p-1)} u_{2n-2} : a \geq 0, b > 0] \\
&\otimes \mathbb{F}_p[\beta Q^a_{p-1}\beta Q^b_{2(p-1)} u_{2n-2} : a, b > 0,]
\end{aligned}$$

For the completion of the classical type, we turn to the mod 2 cases of $Sp(n)$ and $SO(n)$.

Theorem 4. [5] *As an algebra,*

$$\begin{aligned} H_*(\Omega^3_0 Sp(n);\mathbb{F}_2) = &\mathbb{F}_2[Q^a_1 Q^b_2[1] * [-2^{a+b}] : a \geq 0, b \geq 0] \\ &\otimes \mathbb{F}_2[Q^a_1 Q^b_2 z_{4m} : 1 \leq m \leq n-1,\ a \geq 0, b \geq 0]. \end{aligned}$$

Since $Spin(n)$ is a double covering space of $SO(n)$ for $n \geq 3$, $\Omega^3 Spin(n) \backsimeq \Omega^3 SO(n)$. So we can consider the $Spin(n)$ instead of $SO(n)$.

Theorem 5. [6] *As an algebra*
$H_*(\Omega^3_0 Spin 8n;\mathbb{F}_2)$, *$n>0$, is*

$$\mathbb{F}_2[x_{4k} : 1\le k\le n-1]\otimes\mathbb{F}_2[Q_1^a y_{8n+8k-3} : a\ge 0, 0\le k\le n-1]\bigotimes$$
$$\mathbb{F}_2[Q_1^aQ_2^b x_{4n+4k} : a,b\ge 0, 0\le k\le n-1]\bigotimes$$
$$\mathbb{F}_2[Q_1^aQ_2^b z_{8n-4+2k} : a,b\ge 0, 0\le k\le 4n-2 \text{ and } k\not\equiv 3 \mod 4]$$

$H_*(\Omega^3_0 Spin(8n+1);\mathbb{F}_2$, *$n>0$, is*

$$\mathbb{F}_2[x_{4k} : 1\le k\le n-1]\otimes\mathbb{F}_2[Q_1^a y_{8n+8k-3} : a\ge 0, 0\le k\le n-1]\bigotimes$$
$$\mathbb{F}_2[Q_1^aQ_2^b x_{4n+4k} : a,b\ge 0, 0\le k\le n-1]\bigotimes$$
$$\mathbb{F}_2[Q_1^aQ_2^b z_{8n-2+2k} : a,b\ge 0, 0\le k\le 4n-1 \text{ and } k\not\equiv 2 \mod 4]$$

$H_*(\Omega^3_0 Spin(8n+2);\mathbb{F}_2)$, *$n>0$, is*

$$\mathbb{F}_2[x_{4k} : 1\le k\le n-1)\otimes\mathbb{F}_2[Q_1^a y_{8n+8k+5} : a\ge 0, 0\le k\le n-2]\bigotimes$$
$$\mathbb{F}_2[Q_1^aQ_2^b x_{4n+4k} : a,b\ge 0, 0\le k\le n-1]\bigotimes$$
$$\mathbb{F}_2[Q_1^aQ_2^b z_{8n+2k} : a,b\ge 0, 0\le k\le 4n-2 \text{ and } k\not\equiv 1 \mod 4]$$
$$\otimes\mathbb{F}_2[Q_2^a z_{8n-2} : a\ge 0]\otimes\mathbb{F}_2[Q_1^aQ_3^b y_{16n-3} : a,b\ge 0]$$

$H_*(\Omega^3_0 Spin(8n+3);\mathbb{F}_2)$ *is*

$$\mathbb{F}_2[x_{4k} : 1\le k\le n-1]\otimes\mathbb{F}_2[Q_1^a y_{8n+8k+5} : a\ge 0, 0\le k\le n-1]\bigotimes$$
$$\mathbb{F}_2[Q_1^aQ_2^b x_{4n+4k} : a,b\ge 0, 0\le k\le n-1]\bigotimes$$
$$\mathbb{F}_2[Q_1^aQ_2^b z_{8n+2k} : a,b\ge 0, 0\le k\le 4n \text{ and } k\not\equiv 1 \mod 4]$$

$H_*(\Omega^3_0 Spin(8n+4);\mathbb{F}_2)$ *is*

$$\mathbb{F}_2[x_{4k} : 1\le k\le n-1]\otimes\mathbb{F}_2[Q_1^a y_{8n+8k+5} : a\ge 0, 0\le k\le n-1]\bigotimes$$
$$\mathbb{F}_2[Q_1^aQ_2^b x_{4n+4k} : a,b\ge 0, 0\le k\le n)\bigotimes$$
$$\mathbb{F}_2[Q_1^aQ_2^b z_{8n+2k} : a,b\ge 0, 0\le k\le 4n \text{ and } k\not\equiv 1 \mod 4]$$

$H_*(\Omega^3_0 Spin(8n+5);\mathbb{F}_2)$ *is*

$$\mathbb{F}_2[x_{4k} : 1\le k\le n-1]\otimes\mathbb{F}_2[Q_1^a y_{8n+8k+5} : a\ge 0, 0\le k\le n-1]\bigotimes$$
$$\mathbb{F}_2[Q_1^aQ_2^b x_{4n+4k} : a,b\ge 0, 0\le k\le n]\bigotimes$$
$$\mathbb{F}_2[Q_1^aQ_2^b z_{8n+4+2k} : a,b\ge 0, 0\le k\le 4n \text{ and } k\not\equiv 3 \mod 4]$$

$H_*(\Omega^3_0 Spin(8n+6);\mathbb{F}_2)$ *is*

$$\mathbb{F}_2[x_{4k} : 1\le k\le n)\otimes\mathbb{F}_2[Q_1^a y_{8n+8k+5} : a\ge 0, 0\le k\le n-1]\bigotimes$$
$$\mathbb{F}_2[Q_1^aQ_2^b x_{4n+4k+4} : a,b\ge 0, 0\le k\le n-1]\bigotimes$$
$$\mathbb{F}_2[Q_1^aQ_2^b z_{8n+4+2k} : a,b\ge 0, 0\le k\le 4n \text{ and } k\not\equiv 3 \mod 4]$$
$$\otimes\mathbb{F}_2[Q_2^{a+1} x_{4n} : a\ge 0]\otimes\mathbb{F}_2[Q_1^aQ_3^b y_{16n+5} : a,b\ge 0]$$

$H_*(\Omega^3_0 Spin(8n+7);\mathbb{F}_2)$ *is*

$$\mathbb{F}_2[x_{4k} : 1\le k\le n]\otimes\mathbb{F}_2[Q_1^a y_{8n+8k+5} : a\ge 0, 0\le k\le n]\bigotimes$$
$$\mathbb{F}_2[Q_1^aQ_2^b x_{4n+4k+4} : a,b\ge 0, 0\le k\le n-1]\bigotimes$$
$$\mathbb{F}_2[Q_1^aQ_2^b z_{8n+4+2k} : a,b\ge 0, 0\le k\le 4n+2 \text{ and } k\not\equiv 3 \mod 4]$$

where

$$\mathbb{F}_2[Q_1^a Q_2^b x_0] = \mathbb{F}_2[Q_1^a Q_2^b[1] * [-2^{a+b}] : a, b \geq 0)$$
$$\mathbb{F}_2[Q_1^a Q_2^b z_0] = \mathbb{F}_2[Q_1^a Q_2^b[1] * [-2^{a+b}] : a, b \geq 0]$$
$$\mathbb{F}_2[Q_2^{a+1} x_0] = \mathbb{F}_2[Q_2^a(Q_2[1] * [-2]) : a \geq 0]$$

The exceptional Lie groups when localized at p splits as following [11]:

G_2	$p=3$	$B_2(3,11)$
	$p=5$	$B(3,11)$
	$p>5$	$S^3 \times S^{11}$
F_4	$p=5$	$B(3,11) \times B(15,23)$
	$p=7$	$B(3,15) \times B(11,23)$
	$p=11$	$B(3,23) \times S^{11} \times S^{15}$
	$p>11$	$S^3 \times S^{11} \times S^{15} \times S^{23}$
E_6	$p=5$	$B(3,11) \times B(15,23) \times B(9,17)$
	$p=7$	$B(3,15) \times B(11,23) \times S^9 \times S^{17}$
	$p=11$	$B(3,23) \times S^9 \times S^{11} \times S^{15} \times S^{17}$
	$p>11$	$S^3 \times S^9 \times S^{11} \times S^{15} \times S^{17} \times S^{23}$
E_7	$p=5$	$B(3,11,19,27,35) \times B(15,23)$
	$p=7$	$B(3,15,27) \times B(11,23,35) \times S^{19}$
	$p=11$	$B(3,23) \times B(15,35) \times S^{11} \times S^{19} \times S^{27}$
	$p=13$	$B(3,27) \times B(11,35) \times S^{15} \times S^{19} \times S^{23}$
	$p=17$	$B(3,35) \times S^{11} \times S^{15} \times S^{19} \times S^{23} \times S^{27}$
	$p>17$	$S^3 \times S^{11} \times S^{15} \times S^{19} \times S^{23} \times S^{27} \times S^{35}$
E_8	$p=7$	$B(3,15,27,39) \times B(23,35,47,59)$
	$p=11$	$B(3,23) \times B(15,35) \times B(27,47) \times B(39,59)$
	$p=13$	$B(3,27) \times B(15,39) \times B(23,47) \times B(35,39)$
	$p=17$	$B(3,35) \times B(15,47) \times B(27,59) \times S^{23} \times S^{39}$
	$p=19$	$B(3,39) \times B(23,59) \times S^{15} \times S^{27} \times S^{35} \times S^{47}$
	$p=23$	$B(3,47) \times B(15,59) \times S^{23} \times S^{27} \times S^{35} \times S^{39}$
	$p=29$	$B(3,59) \times S^{15} \times S^{23} \times S^{27} \times S^{35} \times S^{39} \times S^{47}$
	$p>29$	$S^3 \times S^{15} \times S^{23} \times S^{27} \times S^{35} \times S^{39} \times S^{47} \times S^{59}$

The space $B(2n_1+1, \ldots, 2n_r+1)$ is build up from fibrations involving p–local spheres of the indicated dimensions, and is equivalent to a direct factor of the p–localization of $SU(n+p)/SU(n)$.

$B(2n+1, 2n+2p-1)$ equivalent to a direct factor of the p–localization of $SU(n+p)/SU(n)$ and its cohomology ring of $B(2n+1, 2n+2p-1)$ is

$$H^*(B(2n+1, 2n+2p-1); \mathbb{F}_p) = E(x_{2n+1}, x_{2n+2p-1})$$

with $\mathcal{P}^1 x_{2n+1} = x_{2n+2p-1}$.

Theorem 6. [7] *For an odd prime p , as an algebra the homologies of $\Omega^3 B(2n+1, 2n+2p-1)$ are given by:*

$$H_*(\Omega^3 B(2n+1, 2n+2p-1); \mathbb{F}_p) = H_*(\Omega^3 S^{2n+1}; \mathbb{F}_p)$$
$$\otimes H_*(\Omega^3 S^{2n+2p-1}; \mathbb{F}_p) \quad \textit{for } n > 1,$$

$$H_*(\Omega_0^3 B(3, 2p+1); \mathbb{F}_p) = \mathbb{F}_p[Q^a_{2(p-1)}(Q_{2(p-1)}[1] * [-p]) : a \geq 0]$$
$$\otimes E(Q^a_{p-1} Q^b_{3(p-1)} u_{2p^2-3} : a \geq 0, b \geq 0)$$
$$\otimes \mathbb{F}_p[\beta Q^a_{p-1} Q^b_{3(p-1)} u_{2p^2-3} : a \geq 0, b > 0].$$

where $\Omega_0^3 B(3, 2p+1)$ be the zero component of $\Omega^3 B(3, 2p+1)$.

From now on we denote $H_*(\Omega^3 S^n; \mathbb{F}_p)$ by $\Omega_3(n)$ and $\otimes_{k=1}^r H_*(\Omega^3 S^{n_k}; \mathbb{F}_p)$ by $\Omega_3(n_1, \cdots, n_r)$.

Theorem 7. [7] *As an algebra,*

$$\begin{aligned} H_*(\Omega_0^3 F_4; \mathbb{F}_2) =& \mathbb{F}_2[Q_1^a u_5 : a \geq 0] \\ & \otimes \Omega_3(9, 11, 15, 23) \\ H_*(\Omega_0^3 F_4; \mathbb{F}_3) =& \mathbb{F}_3[Q_4^a u_{16} : a \geq 0] \\ & \otimes E(Q_2^a Q_6^b u_{15} : a \geq 0, b \geq 0) \\ & \otimes \mathbb{F}_3[\beta Q_2^a Q_6^b u_{15} : a > 0, b \geq 0] \\ & \otimes E(Q_2^a \beta Q_4^b u_i : i = 8, 12, 20, a \geq 0, b > 0) \\ & \otimes \Omega_3(11, 15, 23). \end{aligned}$$

Theorem 8. [8] *The homology of $\Omega_0^3 E_6$ is as follows:*

$$\begin{aligned} (a) H_*(\Omega_0^3 E_6; \mathbb{F}_2) \ =& \ \mathbb{F}_2[Q_2^a u_6 : a \geq 0] \otimes \mathbb{F}_2[Q_1^a \beta_3 Q_2^2 u_6 : a \geq 0, b \geq 0] \\ \otimes & \ \mathbb{F}_2[Q_1^a Q_3^b u_{61} : a \geq 0, b \geq 0] \otimes \Omega_3(11, 15, 23), \\ & \ \textit{where } \beta_3 Q_2^{a+3} u_6 = Q_3^a u_{61} \textit{ for } a \geq 0. \end{aligned}$$

$$\begin{aligned} (b) H_*(\Omega_0^3 E_6; \mathbb{F}_3) =& \otimes \mathbb{F}_3[Q_4^a u_{16} : a \geq 0] \otimes E(Q_2^a \beta Q_4^b u_{16} : a \geq 0, b \geq 0) \\ \otimes & \ \mathbb{F}_3[\beta Q_2^a \beta Q_4^b u_{16} : a > 0, b \geq 0] \otimes \Omega_3(9, 11, 15, 17, 23). \end{aligned}$$

Theorem 9. [8] *The homology of $\Omega_0^3 E_7$ is as follows:*

$$(a) H_*(\Omega_0^3 E_7; \mathbb{F}_2) = \mathbb{F}_2[Q_1^a u_{29} : a \geq 0] \otimes \Omega_3(11, 15, 19, 23, 27, 33, 35)$$

$$\begin{aligned} (b) H_*(\Omega_0^3 E_7; \mathbb{F}_3) =& \mathbb{F}_3[Q_4^a u_{16} : a \geq 0] \otimes E(Q_2^a \beta_2 Q_4^b u_{16} : a \geq 0, b > 0) \\ & \otimes \mathbb{F}_3[\beta Q_2^a \beta_2 Q_4^b u_{16} : a > 0, b > 0] \otimes \Omega_3(11, 15, 23, 27, 35). \end{aligned}$$

Theorem 10. [8] *The homology of $\Omega_0^3 E_8$ is as follows:*

$$\begin{aligned} (a) H_*(\Omega_0^3 E_8; \mathbb{F}_2) =& \mathbb{F}_2[u_{12}] \otimes \mathbb{F}_2[Q_1^a u_i : a \geq 0, i = 29, 53] \\ & \otimes \Omega_3(23, 27, 33, 35, 39, 47, 57, 59) \end{aligned}$$

$$\begin{aligned} (b) H_*(\Omega_0^3 E_8; \mathbb{F}_3) =& \mathbb{F}_3[Q_4^a u_{52} : a \geq 0] \otimes E(Q_2^a \beta Q_4^b u_{52} : a \geq 0, b > 0) \\ & \otimes \mathbb{F}_3[\beta Q_2^a \beta Q_4^b u_{52} : a > 0, b > 0] \\ & \otimes \Omega_3(15, 23, 27, 35, 39, 47, 59) \end{aligned}$$

$$\begin{aligned} (c) H_*(\Omega_0^3 E_8; \mathbb{F}_5) =& E(Q_4^a u_{47} : a \geq 0) \otimes \mathbb{F}_5[\beta Q_4^a u_{47} : a > 0] \\ & \otimes \mathbb{F}_5[Q_8^a u_{48} : a \geq 0] \otimes E(Q_4^a \beta Q_8^b u_{48} : a \geq 0, b > 0) \\ & \otimes \mathbb{F}_5[\beta Q_4^a \beta Q_8^b u_{48} : a > 0, b > 0] \\ & \otimes \Omega_3(15, 23, 27, 35, 39, 47, 59). \end{aligned}$$

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