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SOME RECENT ADVANCES IN HYPERSPACE TOPOLOGIES AND SELECTION THEORIES

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ABSTRACT. In the first part of this paper, we shall introduce hypertopologies on the collection of some subsets of a topological space. In the second part, we shall apply these topologies to such topics as set-valued mappings.

1. Introduction

There has been many impressive achievements in hyperspace topologies and selection theories (see [2],[29], and [30] for a survey). New ideas and new names recently appeared. I am not trying to present here a comprehensive survey of all major developments. Instead, I have chosen to write about results in which I was interested for quite a while. Thus, it is a very subjective survey.

By a *hyperspace* of a T_1 topological space (X, τ) we mean the set $CL(X)$ of the nonempty closed subsets of X, endowed with a topology $\mathcal{T}$ such that the function $i : (X, \tau) \to (CL(X), \mathcal{T})$ defined as $i(x) = \{x\}$ is a homeomorphism onto its image. Since the beginning of this century some *hyperspace topologies*, also called *hypertopologies*, have been introduced and developed ; in particular, the Hausdorff metric and Vietoris topologies. These two topologies are very fine, at least in view of some applications. It is a remarkable fact that the most important hyperspace topologies arise as topologies induced by families of geometric set functionals. We give a particular attension to the interplay between hyperspaces and geometrical functional analysis, and to the convergence of lower semicontinuous functions as associated with their *epigraphs*. Although hyperspace topologies and related set convergence notions have been investigated since the beginning of the century, the approach we take to the subject reflects decisive modern contributions by mathematicians whose primary research interests lie outside general topology. The revival of the subject stems from work of Robert Wijsman[33] in the middle of 1960' s, and its development over the next fifteen years was to a large extent in the hands of U. Mosco, R. Wets, H. Attouch, and their associates. This new approach was developed for the most part in North America, France, and Italy.

This increasing interest is owing to usefulness of these in different fields of applications such as probability, statistics or variational problems for instance. It also explains the effort in understanding their structure, common features and general patterns in order to find a common description for them. The papers [3], [31,32] or

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more recently [22], are partially or completely devoted to this goal, offering various possibilities of generalization.

This survey paper is also dedicated to the theory of continuous selections of set-valued mappings, a classical area of mathematics as well as an area which has been intensively developing in recent decades and has found various applications in general topology, theory of absolute retracts and infinite dimensional manifolds, geometric topology, fixed-point theory, functional and convex analysis, and other branches of modern mathematics.

The fundamental results in this theory stemmed from the mid 1950's by E. Michael [23,24,25,26].

As classical selection theorems, there are four main selection theorems: zero-dimensional, convex-valued, compact-valued, and finite-dimensional theorems.

This paper is focused only on a convex-valued selection theorem.

We will first look at the theory of selection based on Michael's papers from the mid-50s and then consider some recent advances in selection theory related to hypertopologies[14,15,16,17]. Let X and Y be topological spaces, and let 2^Y be the family of nonempty subsets of Y. A continuous mapping $f : X \to Y$ is a *selection* for a set-valued mapping $\Phi : X \to 2^Y$ if $f(x) \in \Phi(x)$ for every $x \in X$. Most of the classical Michael-type selection theorems establish that the existence of continuous selections for lower semi-continuous mappings $\Phi : X \to CL(X)$ is actually equivalent to some separation properties (like paracompactness, collectionwise normality, normality, etc.) of X. For undefined terminology, we follow [2] and [12].

2. Hypertopologies

Let us first describe notations we are going to deal with. For a toplogical space X and $E \subset X$ write $E^- = \{A \in CL(X) : A \cap E \neq \emptyset\}, E^+ = \{A \in CL(X) : A \subset E\}$; further if $(X,\mathcal{U})$ is a uniform space, put $E^{++} = \{A \in CL(X) : \exists U \in \mathcal{U}$ with $U[A] \subset E\}$, where $U[A] = \{x \in X : \exists a \in A$ with $(x, a) \in U\}$.

There are three types of hypertopologies : the *hit-and-miss*, the *proximal hit-and-miss*, and *the weak topologies generated by gap and excess functionals* on $CL(X)$, respectively.

Hit-and-miss topology: The abstract hit-and-miss-topology on $CL(X)$ has as a subbase all sets of the form V^-, where V is open in X, plus all sets of the form $(B^c)^+$, where $B^c = X \backslash B$ and B ranges over a fixed nonempty subfamily $\triangle \subset CL(X)$. It was studied in [2], [5,6], [10], [18], [34,35,36]. The well-known prototypes of hit-and-miss topologies are the *Vietoris topology*, with $\triangle = CL(X)$([2], [23]) and the *Fell topology*, with $\triangle$ =nonempty closed compact subsets of X([2], [13], [21]).

Proximal hit-and-miss topology: If $(X,\mathcal{U})$ is a uniform space and $(B^c)^+$ is replaced by $(B^c)^{++}$ in the above definition, we get the proximal hit-and-miss topology (or hit-and-far topology). It was studied in [2], [5,6], [10], and [36]. For example, among its useful prototypes we can find the *proximal topology*, with $\triangle = CL(X)$([4]) or the *ball-proximal topology*, with $\triangle$ = closed proper balls in a metric space X([9], [19], [36]).

Weak topologies generated by gap and excess functionals: In a metric space (X, d), we define the *distance* functional $d(x, A) = inf\{d(x, a) : a \in A\}$ $(x \in X, \emptyset \neq A \subset X)$, the *gap* functional $D(A, B) = inf\{d(a, B) : a \in A\}$ $(A, B \subset X)$, and the *excess* functional $e(A, B) = sup\{d(a, B) : a \in A\}$ $(A, B \subset X)$. Then the so-called *weak hypertopologies* (or *initial topologies*) on $CL(X)$ are defined as the weak topologies generated by gap (in particular, distance) and excess functionals, where one of the set arguments of $D(A, B)$ and $e(A, B)$, respectively ranges over given subfamilies of $CL(X)$. A good reference about these topologies is [2], [3], [20], or [37]. As a prototype of weak topologies we should mention the *Wijsman topology*, which is the weak topology generated by the distance functionals viewed as functionals of set argument ([2], [19], [36]).

2.1 Theorem. *[4] Let X be a metrizable space, and let $\mathcal{D}$ denote the set of compatible metrics for X. Then the Vietoris topology on $CL(X)$ is the weak topology determined by the family of distance functionals $\{d(x,\cdot) : x \in X, d \in \mathcal{D}\}$.*

2.2 Theorem. *[2] Let (X, d) be a metric space and let $\mathcal{D}_d$ denote the set of metrics that are uniformly equivalent to d(i.e., they determine the same uniformity). Then weak topology on $CL(X)$ determined by the family of distance functionals $\{\rho(x,\cdot) : x \in X, \rho \in \mathcal{D}_d\}$ coincides with the proximal topology determined by d.*

2.3 Theorem. *[2] Let (X, d) be a metric space. Then the proximal topology is the weak topology on $CL(X)$ determined by the family of excess functionals $\{e_d(\cdot, B) : B \in CL(X)\}$.*

3. Selection Theory

Let X and Y be topological spaces, and 2^Y be the family of nonempty subsets of Y. A mappping $\Phi : X \to 2^Y$ is called a *set-valued mapping*. A *selection* for $\Phi : X \to 2^Y$ is a continuous map $f : X \to Y$ such that $f(x) \in \Phi(x)$ for every $x \in X$.

A set-valued mapping $\Phi : X \to 2^Y$ is called *lower semi-continuous* (respectively, *upper semi-continuous*) or l.s.c. (respectively, u.s.c.) if for every open subset V of Y,

$$\Phi^{-1}(V) = \{x \in X : \Phi(x) \cap V \neq \emptyset\}$$

$$(\text{respectively}, \Phi^{\#}(V) = \{x \in X : \Phi(x) \subset V\})$$

is an open subset of X.

Let (Y, d) be a metric space and let for $S \in 2^Y$ and $\varepsilon > 0$, $B^d_\varepsilon(S)$ denote $\{y \in Y : d(y, S) < \varepsilon\}$.

A mapping $\Phi : X \to 2^Y$ is d-l.s.c. (respectively, d-u.s.c.) if , given $\varepsilon > 0$, every $x \in X$ admits a neighbourhood U such that for every $u \in U$,

$$\Phi(x) \subset B^d_\varepsilon(\Phi(u)) \ (respectively, \Phi(u) \subset B^d_\varepsilon(\Phi(x)).$$

The following new important notions of continuity of set-valued mappings arise via combinations of the above versions of l.s.c. and u.s.c.. Namely, a set-valued mapping Φ is *continuous* if it is both l.s.c. and u.s.c.; Φ is *d-continuous* if it is d-l.s.c. and d-u.s.c.; and finally Φ is *d-proximal continuous* (see [16])if it is both l.s.c. and d-u.s.c.. A continuous Φ is not necessarily d-continuous and vice versa(see, e.g., [16, Proposition 2.5]), while every continuous or d-continuous Φ is certainly d-proximal continuous. But, there are d-proximal continuous mappings Φ which are neither continuous nor d-continuous (see [16]). In view of that, we shall henceforth restrict our attention only to d-proximal continuity. This property, however, depends on the metric d on the range Y. To overcome this, following [16], we shall say that $\Phi : X \to 2^Y$ is *proximal continuous*, where Y is metrizable, if there exists a compatible metric d on Y such that Φ is d-proximal continuous.

To begin with, we give a very important example which is a prototype for selection theory.

3.1 Example [24]. *Let $\Phi : X \to 2^Y$, $A \subset X$, and let $g : A \to Y$ be a selection for $\Phi|A$. Define $\Phi_g : X \to 2^Y$ by*

$$\Phi_g(x) = \begin{cases} \{g(x)\}, & \text{if } x \in A \\ \Phi(x), & \text{if } x \in X - A \end{cases}$$

If Φ is l.s.c., A is closed, and g is continuous, then Φ_g is also l.s.c..

Moreover, $f : X \to Y$ is a selection for Φ_g if and only if f is a selection for Φ which extends g.

Thus, regarding l.s.c. mappings, the problem of *extending* a selection is effectively reduced to the simpler problem of merely *finding* one.

However, this fails for continuous or d-continuous Φ. For properly situated subsets A of X and continuous selections g for $\Phi|A$, there is no hope Φ_g to remain (d)-l.s.c. and (d)-u.s.c. simultaneously.

An extension problem is one of the important branch in general topology. As we will see, selection theories are very closely related to extension problems. We consider a following general extension problem in general topology:

General Question: Let X and Y be two topological spaces with $A \subset X$ closed, and let $f : A \to Y$ be continuous. Under what conditions on X and Y, does f have a continuous extension over X (or at least over some open $U \supset A$) ?

3.2 Theorem. *(Tietze's extension theorem) Let X be a normal space. Then for every closed $A \subset X$ and every continuous function $f : A \to \mathbb{R}$, there exists a continuous extension of f over X.*

We are going to generalize Tietze's extension theorem by use of a new point of view due to E. Michael[24,25,26]. He considered additional requirements on f as follows:

For every $x \in X$, $f(x)$ must be an element of a "pre-assigned" subset of Y. This new problem is called the *selection problem.*

3.3 Theorem. *(Theorem $3.2''$ in [24]) For a T_1 space X the following are equivalent: (a) X is paracompact (b) If Y is a Banach space, then every lower semi-continuous set-valued map for X to the family of non-empty, closed, convex subsets of Y admits a selection.*

This theorem characterizes paracompactness using continuous functions rather than covering properties. Since the continuous functions are relatively easier to handle covering properties, the above Michael's theorem is important and applicable to certain proofs. Collectionwise normality is another strengthening of normality, weaker than paracompactness.

We can also give a characterization of collectionwise normality using continuous functions as in Michael's paper (see [24]). The proof of the following fact is basically due to H. Dowker [11], but we can give a new and short proof using selection theory. This is also a generalization of Tietze's extension theorem.

3.4 Theorem. *A T_1-space X is collectionwise normal if and only if every continuous mapping of every closed subset of X into a Banach space can be continuously extened over X.*

Recall that a space X is *metacompact* (or *weakly paracompact*) if every open cover of X has a point-finite open refinement. Every countably compact metacompact space is compact. M. Choban [7,8] characterized metacompactness, which is weaker than paracompactness, using a continuous function. The following theorem is so called a *selection extension property.*

3.5 Theorem. *[24] Let X and Y be topological spaces and $\mathcal{S} \subset 2^Y$ be a collection containing all single-ton sets. Then the following are equivalent.*

(i) If $\Phi : X \to \mathcal{S}$ is a l.s.c. set-valued mapping, then for every closed $A \subset X$, each selection for $\Phi|A$ can be extended to a selection for Φ. (ii) Every l.s.c. set-valued mapping $\Phi : X \to \mathcal{S}$ has a selection.

Theorem 3.5 reduces the problem which deals with extending a selection to the simpler problem of merely finding one.

If $A \subset X$ is closed, then $\Phi : X \to 2^Y$ has the *selection extension property*, or SEP, at A if every selection for $\Phi|A$ extends to a selection for Φ; if Φ has the SEP at every closed $A \subset X$, then we simply say that Φ has the SEP.

3.6 Question. *[24] Under what conditions on X, $A \subset X$, and $\Phi : X \to 2^Y$, can every selection for $\Phi|A$ be extended to a selection for Φ, or at least for $\Phi|U$ for some open $U \supset A$?*

More questions related to selection problems can be found in [28].

One of the results related to this question is the following theorem which strengthens some known selection theorems for set-valued mappings $\Phi : X \to 2^Y$ by eliminating all assumptions, except lower semi-continuity, an arbitrary countable subset of X.

3.7 Theorem. *[27] If X is countable and regular, Y is first-countable, and $\Phi : X \to 2^Y$ is l.s.c., then Φ has SEP.*

As we have shown above, the existence of continuous selections for lower semi-continuous set-valued mappings of X into the family of nonempty, closed, convex subsets of a Banach space Y implies some separation properties (like paracompactness, colllectionwise normality, etc.) of X. The following result shows that we can dispense with the separation properties of X strengthening the restriction on the continuity of Φ.

3.8 Theorem. *[16] Let X be a topological space and let Y be a Banach space. Then every d-proximal continuous mapping Φ from X to the family of nonempty, closed, convex subsets of Y admits a selection.*

In Theorem 3.8, we can weaken the restriction on the continuity of Φ provided Y is reflexive (see [17]).

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