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LERF AND LOW DIMENSIONAL TOPOLOGY

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ABSTRACT. we describe the connection between LERF and low dimensional topology and introduce both LERF 3-manifold groups and non-LERF 3-manifold groups.

A subgroup S of a group G is called to *subgroup separable* if S is the intersection of the subgroups of finite index containing it (equivalently, for each $g \in G \backslash S$, G has a subgroup G_1 of finite index which contains S but not g). A group G is called *locally extended residually finite* (LERF) if every finitely generated subgroup is subgroup separable. Similarly, a group G is called to be *residually finite* if the intersection of the subgroups of finite index is trivial. Note LERF condition is a little stronger condition than residual finite condition which is very useful in combinatorial group theory. For instance, every finitely generated residually finite group is Hopfian.

LERF- groups are of interest in low dimensional topology as well as in combinatorial group theory. A subgroup S of the fundamental group of a manifold M is called *geometric* in $\pi_1(M)$ if there is a codimension zero submanifold $N \subset M$ such that the natural map $\pi_1(N) \hookrightarrow \pi_1(M)$ is an isomorphism onto a conjugate of S in $\pi_1(M)$. The subgroup S is *almost geometric* if there is a finite cover M_1 such that S is geometric in $\pi_1(M_1)$. It is an interesting question in low dimensional topology whether the fundamental group of a manifold is almost geometric. P. Scott shows in [8, 9] that the fundamental group of a manifold being LERF is closely related to each finitely generated subgroup of it being almost geometric in 2 and 3 dimensional manifold cases. Hence it is reasonable to ask whether a given manifold is LERF.

It is shown in [8, 9] that the fundamental groups of closed surfaces and Seifert fibered spaces are LERF. On the other hand, non-LERF 3-manifold groups are given in [6]. It is the fundamental group of a torus sum of two Seifert fibered spaces. Hence 3-manifold groups are not LERF in general. However it is still not known whether 3-dimensional hyperbolic manifold groups are LERF. P. Scott suggested in [8] a method for proving that a given hyperbolic 3-manifold groups are LERF.

The material presented in this paper is drawn out of the papers [6], [8] and [9]. Most of 3-manifold terminology can be found in [5].

1. LERF and Almost Geometric

We start by stating a few lemma on LERF and its connection to the almost geometric condition.

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Lemma 1.1. *If G is residually finite or LERF, then any subgroup and finite extension of it have the same property.*

Lemma 1.2. *Let X be a Hausdorff topological space with a regular covering $\tilde{X}$ and a covering group G. Then G is LERF if and only if given a finitely generated subgroup S of G and a compact subset C of $\tilde{X}/S$, there is a finite covering X_1 of X such that the projection $\tilde{X}/S$ factors through X_1 and C projects homeomorphically into X_1.*

The results in [7] and [8] and Lemma 1.2 results in the following lemma.

Lemma 1.3. *Let M be a manifold of dimension 2 or 3. Then $\pi_1(M)$ is LERF if and only if given a finitely generated subgroup S of $\pi_1(M)$ and $g \in \pi_1(M) \setminus S$, there is a finite covering M_1 of M such that $\pi_1(M_1)$ contains S but not g. Further, S is geometric in M_1.*

It was first proved by Hall [?] that non-closed subgroups (i.e. free groups) are LERF. The following theorem is equivalent to Hall's result in view of Lemma 1.3.

Theorem 1.1. *Let F be a non-closed surface, and let $g \in \pi_1(F) \setminus S$. Then there is a finite covering F_1 of F such that $\pi_1(F_1)$ contains S but not g and S is geometric in F_1. Further S is a free factor of $\pi_1(F_1)$.*

We now discuss the closed surface group case. Let F be a closed surface. If F is either the sphere or the projective plane, then $\pi_1(F)$ is finite, and so it is LERF. It can also be easily proved that $\mathbb{Z} \times \mathbb{Z}$ is residually finite. Since the torus covers the Klein-bottle, the fundamental groups of the torus and Klein-bottle are LERF. Any other closed surface covers the closed non-orientable surface $F_{-1}$with Euler characteristic -1 (i.e. the connected sum of the torus and projective plane).

It is known that there is a regular pentagon P in $\mathbb{H}^2$ with all its vertex angles of P equal to $\pi/2$. Let , denote the group of isometries of $\mathbb{H}^2$ generated by the reflections in the sides of P. It is not hard to see that , has a subgroup isomorphic to $\pi_1(F_{-1})$. Hence in order to show that every Fuchsian group and every surface group is LERF, it suffices to show that $\pi_1(F_{-1})$ is LERF by Lemma 1.1.

Theorem 1.2. $\pi_1(F_{-1})$ *is LERF.*

Proof. Let G be the group $\pi_1(F_{-1})$. Suppose we are given a finitely generated subgroup S of G and a compact subset C of $\mathbb{H}^2/S$. Let $p : \mathbb{H}^2 \to \mathbb{H}^2/S$ denote the covering map. Let D be a compact set in $\mathbb{H}^2$ such that $p(D) = C$. It suffices to show that there is a subgroup G_1 of finite index in G such that G_1 contains S and, in addition, if an element g of G_1 has gD meeting D then g lies in S.

As C is a compact subset of $\mathbb{H}^2/S$ and as $\pi_1(\mathbb{H}^2/S) = S$ is finitely generated, there is a compact connected subsurface C_1 of $\mathbb{H}^2/S$, such that C_1 contains C and the natural map $\pi_1(C_1) \to \pi_1(\mathbb{H}^2/S)$ is an isomorphism. Let Y denote $p^{-1}(C_1)$, which is a connected subsurface in $\mathbb{H}^2$. Define $\overline{Y}$ to be the intersection of all closed half spaces in $\mathbb{H}^2$ which contain Y in their interior and are bounded by a line which is a translate of a side of P. Thus $\overline{Y}$ is a convex union of pentagons in $\mathbb{H}^2$. Now Y is invariant under the action of S on $\mathbb{H}^2$, and so it follows that $\overline{Y}$ is S-invariant. It can be shown that $p(\overline{Y})$ is compact. Let X be a fundamental region in $\overline{Y}$ for the action of S.

We define H_2 to be the group of isometries of $\mathbb{H}^2$ generated by the reflections in the sides of $\overline{Y}$. $\overline{Y}$ is a fundamental region for H_2, because all the vertex angles of

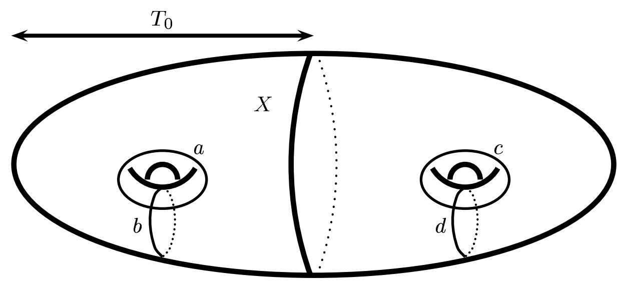

FIGURE 1

$\overline{Y}$ are equal to $\pi/2$. Define H_1 to be the group of isometries of $\mathbb{H}^2$ generated by H_2 and S. The group $G_1 = G \cap H_1$ has all the required properties. □

Using the following lemma and the fact that surface groups are LERF, it can be shown that the fundamental group of a compact Seifert fibered space is LERF.

Lemma 1.4. *Let H denote the fundamental group of a compact orientable surface F and let G be a central extension of an infinite cyclic group J by H. Let S be a finitely generated subgroup of G and let J_1 be a subgroup of finite index k in J such that J_1 contains $S \cap J$. Then G has a subgroup G_1 of finite index such that G_1 contains S and $G_1 \cap J = J_1$.*

Theorem 1.3. *The fundamental group of a compact Seifert fibered space is LERF.*

2. Non-LERF 3-manifold groups

Let B be the group whose presentation is

$$< a, b, t \mid tbt^{-1} = b, tat^{-1} = ba > .$$

It was shown in [2] that B is not LERF. Note that B is the fundamental group of the mapping torus M_0 of the once puncture torus T_0 by the Dehn twist in the simple closed curve b as shown in the figure 1. Hence B is a non-LERF 3-manifold group. In view of Lemma 1.1, every 3-manifold group containing B as a subgroup is not LERF.

Let T_2 be the orientable closed surface of genus 2 as shown in the Figure 1. Consider the mapping torus $M(\tau_b)$, where τ_b denotes the Dehn twist in the simple closed curve b. Observe that this manifold decomposes along the vertical incompressible torus lying over X into M_0 and $T_0 \times S^1$. This splits the fundamental group of $M(\tau_b)$ as an amalgamated free product $B *_{\mathbb{Z}\times\mathbb{Z}} (F_2 \times \mathbb{Z})$, where F_2 the free group of rank 2. Since B is not LERF, the fundamental group of $M(\tau_b)$ is not LERF. This gives another non-LERF 3-manifold group.

Define a covering of T_2 defined by the map $H_1(T_2) \to \mathbb{Z}_2$ given by $b, c, d \to 0$ and $a \to 1$. This defines a double covering $p : T_3 \to T_2$, where T_3 is the orientable closed surface. The preimage of the curve b is the two curves in T_3, so it follows that there is a lift of the Dehn twist τ_b which we denote by $\tilde{\tau_b}$. It is easy to see that the mapping torus $M(\tilde{\tau_b})$ double covers $M(\tau_b)$. Since $\pi_1(M(\tau_b))$ is not LERF, $\pi_1(M(\tilde{\tau_b}))$ is not LERF by Lemma 1.1.

Now consider the mapping torus $M(\tau_X)$ by the Dehn twist τ_X in the curve X. The map $H_1(T_2) \to \mathbb{Z}_2$ given by $b, d \to 0$ and $a, c \to 1$ defines a double covering map $q : T_3 \to T_2$. The preimage of the curve X is again the pair of the curves as in the previous case. If we denote the lift of τ_X by $\tilde{\tau_X}$, it can be shown that the two mapping tori $M(\tilde{\tau_X})$ and $M(\tilde{\tau_b})$ are homeomorphic to each other. It follows that the group $\pi_1(M(\tau_X))$ is not LERF.

Note that if we split $M(\tau_X)$ along the vertical torus $X \times S^1$, we obtain two copies of $T_0 \times S^1$. $T_0 \times S^1$ is a Seifert fibered space, and so the fundamental group of it is LERF. Hence this example shows us that the free product of two LERF 3-manifold groups amalgamated along the torus group may not be LERF.

In [1], it was shown that the free product of two free groups with cyclic amalgamation is LERF. But a free product of two LERF groups with cyclic amalgamation may not be LERF. In fact, it was shown in [6] that there is an amalgamated free product $(F_2 \times \mathbb{Z}) *_{\mathbb{Z}} (F_3 \times \mathbb{Z})$ which is not LERF, where F_n denotes the free group of rank n, even though $F_n \times \mathbb{Z}$ is LERF.

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