Trends in Mathematics
Information Center for Mathematical Sciences
Volume 1, Number 1, September 1998, Pages 85–90

INVARIANTS OF KNOTS AND LINKS VIA INTEGRAL MATRICES

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ABSTRACT. This is a brief summary of the recent works joint with Dr. Sang Youl Lee, on the Seifert matrices and the (modified) Goeritz matrices of knots and links and their invariants: the Alexander polynomial, the Minkowski unit, the signature, the nullity, and the determinant of a knot and a link. We introduce the relationship between the modified Goeritz matrices of 2-periodic links and those of their factor links and some properties of the 2-parallel version invarants of the Minkowski units, the signature, the nullity, and the square free of the determinant of a knot and a link.

1. Invariants of the Seifert Matrices

Let $\mathbb{R}^3$ denote the oriented Euclidean 3-space and $S^3 = \mathbb{R}^3 \cup \{\infty\}$ the oriented 3-sphere. Throughout this note we will work in the piecewise-linear(PL) category.

A *link of μ-components* is (the image of) an embedding of μ ordered mutually disjoint 1-spheres $S^1_1, S^1_2, \cdots, S^1_\mu$ into S^3(or $\mathbb{R}^3$), denoted by $\ell = k_1 \cup \cdots \cup k_\mu$. In particular, if $\mu = 1$, then it is called a *knot*. If each component k_i of a link ℓ is oriented, then ℓ is called an *oriented link of μ components*. We denote the set of all knots by $\mathcal{K}$ and the set of all links by $\mathcal{L}$. Two links $\ell = k_1 \cup \cdots \cup k_\mu$ and $\ell' = k'_1 \cup \cdots \cup k'_\nu$ in $\mathcal{L}$ are said to be *equivalent*, denoted by $\ell \approx \ell'$, if and only if ℓ and ℓ' are *ambient isotopic*, that is, there exists an isotopy $H : S^3 \times [0,1] \to S^3 \times [0,1]$ such that $H(x,t) = (h_t(x), t)$, where $h_t : S^3 \to S^3$ are homeomorphisms that satisfy $h_0 = id_{S^3}$, the identity mapping on S^3, and $h_1(\ell) = \ell'$.

We denote by $[\ell](resp, [k])$ the equivalence class of a link ℓ (resp, a knot k), called the *link(resp, knot) type of ℓ(resp, k)*. Let $[\mathcal{L}] = \mathcal{L}/\approx = \{[\ell] \mid \ell \in \mathcal{L}\}$(resp, $[\mathcal{K}] = \mathcal{K}/\approx$).

Let $\mathcal{A}$ be a set of objects. A mapping $\rho : [\mathcal{L}] \to \mathcal{A}$ (resp, $\rho : [\mathcal{K}] \to \mathcal{A}$) is called an *invariant of links(respectively, invariant of knots)*. If ρ is in one-to-one correspondance, then ρ is said to be a *complete invariant* of links(respectively, knots).

In 1928, J. W. Alexander[A] first introduced the Alexander polynomial and, in 1934, H. Seifert[Se1] gave a method computing the Alexander polynomial via a Seifert matrix of a knot and also investigated the characterizing properties of the Alexander polynomial of a knot. R. H. Fox ([Fo1,2,3], [BZ]) gave more conceptual description of the Alexander polynomials by using free differential calculus in the group ring of a free group. In this section, we will present briefly the Alexander polynomials of knots and links based on the Seifert matrices and other invariants: the signature and the nullity of a knot and a link.

Key words and phrases. Alexander matrix, Alexander polynomial, Seifert matrix, Goeritz matrix, signature, nullity, Minkowski unit, 2-periodic link, 2-parallel version invariant.

Let ℓ be an oriented knot or link in S^3. An orientable connected surface that spans ℓ is called a *Seifert surface* of ℓ and the *genus of knot or link* ℓ, denoted by $g(\ell)$, is defined to be the minimum of the genus of all Seifert surfaces that span ℓ. Now let ℓ be an oriented link of μ-components and let F be a Seifert surface of ℓ. If the genus of F is equal to $g(F)$, then the Homology group $H_1(F;\mathbb{Z})$ with integral coefficients is a free abelian group with $n = 2g(F) + \mu - 1$ generators. Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ denote oriented simple closed curves that represent a basis for $H_1(F;\mathbb{Z})$. Consider a collar $F \times [0,1]$. For all $i, j \in \{1, 2, \cdots, n\}$, we denote oriented simple closed curves $\alpha_i \times \{0\}$ and $\alpha_j \times \{1\}$ by α_i and $\alpha_j^{\sharp}$, respectively.

Let $M_F(\ell)$ be an $n \times n$ integral matrix defined as follows: If $g(F) = 0$ and $\mu = 1$, then $M_F(\ell)$ is the empty matrix. If not,

$$M_F(\ell) = (Lk(\alpha_i, \alpha_j^{\sharp}))_{1 \le i,j \le n},$$

where $Lk(\alpha_i, \alpha_j^{\sharp})$ denotes the linking number of α_i and $\alpha_j^{\sharp}$. This matrix $M_F(\ell)$ is called the *Seifert matrix* of ℓ associated to F.

Let $\mathcal{M} = \bigcup_{n=0}^{\infty} \mathcal{M}_n$, where $\mathcal{M}_n$ denotes the set of all $n \times n$ integral matrices and let $[M]$ denotes the S-equivalence class of M and let $[\mathcal{M}] := \mathcal{M}/ \approx_S = \{[M] \mid M \in \mathcal{M}\}$. The detailed proofs of Theorem 1, 2, and 3 can be found in Murasugi's book[Mur3] or [Mur 1].

Theorem 1. *The mapping $\mathcal{S} : [\mathcal{L}] \to [\mathcal{M}]$ defined by $\mathcal{S}([\ell]) = [M_F(\ell)]$ is an invariant of knots and links.*

Theorem 2. *The map $\Delta : [\mathcal{L}] \to \mathbb{Z}[t^{\pm\frac{1}{2}}]$ defined by*

$$\Delta([\ell]) = t^{-\frac{n}{2}} det(M_F(\ell) - tM_F(\ell)^T)$$

is an invariant of knots and links, where $n = 2g(F) + \mu - 1$.

The polynomial $\Delta([\ell])$ is called the *(reduced) Alexander polynomial of* ℓ, denoted by $\Delta_\ell(t)$, and the absolute value $|det(M_F(\ell) + M_F(\ell)^T)|$ is also an invariant of a knot(link), called the *determinant of a knot(link).*

Let A be any integral symmetric $n \times n$ matrix. The *signature* and the *nullity* of A are denoted by $\sigma(A)$ and $\mathcal{N}(A)$, respectively. Then

Theorem 3. *(1) The map $\sigma : [\mathcal{L}] \to \mathbb{Z}$ defined by $\sigma([\ell]) = \sigma(M_F(\ell) + M_F(\ell)^T)$ is an invariant of knots and links, called the signature of ℓ and denoted by $\sigma(\ell)$.*

(2) The map $\mathcal{N} : [\mathcal{L}] \to \mathbb{N} \cup \{0\}$ defined by $\mathcal{N}([\ell]) = \mathcal{N}(M_F(\ell) + M_F(\ell)^T)$ is an invariant of knots and links, called the nullity of ℓ and denoted by $\mathcal{N}(\ell)$.

2. Invariants of Goeritz matrices

Let ℓ be an oriented link in S^3 and let L be its link diagram in the plane $\mathbb{R}^2 \subset \mathbb{R}^3 \cong S^3 - \{\infty\}$. Colour the regions of $\mathbb{R}^2 - L$ alternately black and white. Denote the white regions by $X_0, X_1, \cdots, X_w$. We always take the unbounded region to be white and denote it by X_0.

Assign an incidence number $\eta(c) = \pm 1$ to each crossing c of L and divide crossings into two types, *type I* or *type II*(see, [GL]). Let $S(L)$ denote the compact surface with boundary L, more precisely, $S(L)$ is built up out of discs and bands. Each disc lies in $S^2 = \mathbb{R}^2 \cup \{\infty\}$ and is a closed black region less a small neighbourhood of each crossing. Each crossing gives a small half-twisted band.

Let $G'(L)$ be a symmetric integral matrix defined by $G'(L) := [g_{ij}]_{0\leq i,j\leq w}$, where

$$g_{ij} = \begin{cases} -\sum_{c\in C_L(X_i,X_j)} \eta(c) & \text{if } i \neq j, \\ \sum_{c\in C_L(X_i)} \eta(c) & \text{if } i = j \end{cases},$$

where $C_L(X_i, X_j) = \{c \in C(L) | c \text{ is incident to both } X_i \text{ and } X_j\}$ and $C_L(X_i) = \{c \in C(L) | c \text{ is incident to } X_i\}$. The principal minor $G(L) = [g_{ij}]_{1\leq i,j\leq w}$ of $G'(L)$ is called the *Goeritz matrix* associated to the diagram L of ℓ and the quadratic form associated to $G(L)$ is called the *Goeritz form* associated to L of ℓ([Go],[GL]).

In 1933, Goeritz[Go] showed that for odd primes p, the *Minkowski units* $C_p(K)$ of the Goeritz matrix $G(K)$ associated to a knot diagram K of a knot k are invariants of the knot k but not for $C_2(K)$. The absolute value of the *determinant of* $G(K)$ of a kont diagram K is also an invariant of the knot. In 1965, Murasugi[Mur2] generalized the Minkowski units of knots to that of links for all prime numbers including ∞, by using his matrix[Mur1] and proved that the Minkowski units of slice links are always equal to the zero.

H. Seifert ([Se2]) recognized the geometric significance of the Goeritz matrix; if $M_2(k)$ is the 2-fold branched covering space of S^3 branched along a knot k, and $G(K)$ is the Goeritz matrix associated to a knot diagram K of the knot k, then $G(K)$ is a relation matrix for the homology group $H_1(M_2(k))$, and $\pm G(K)^{-1}$ is the matrix of the linking form on $H_1(M_2(k))$. This result was generalized to the case of links by R. H. Kyle[Ky] .

Notice that the signature of the Goeritz matrix of a knot or link are not invariants of the knot or link and the nullity of the Goeritz matrix of a link is not an invariant the link.

Let ℓ be an oriented link and let L be its diagram. Let $C_{II}(L) = \{c_1, c_2, \cdots, c_s\}$ be the set of all type II crossings in L and define the $s \times s$ diagonal matrix $A(L)$ associated to L by $A(L) = diag(-\eta(c_1), -\eta(c_2), \cdots, -\eta(c_s))$. Let $O(L)$ be the $(\beta_0(L) - 1) \times (\beta_0(L) - 1)$ zero matrix, where $\beta_0(L)$ is the number of components of the surface $S(L)$. If $S(L)$ is connected, we set $O(L)$ to be the empty matrix.

Then the *modified Goeritz matrix* $H(L)$ associated to L is defined to be the symmetric matrix given by

$$H(L) = G(L) \oplus A(L) \oplus O(L),$$

where O denotes the zero matrix of appropriate size.

The signature $\sigma(H(L))$ and the nullity $n(H(L))$ of the modified Goeritz matrix $H(L)$ are invariants of knots and links[Tr]. If the surface $S(L)$ is a Seifert matrix of ℓ, then both $A(L)$ and $O(L)$ are empty, so that $H(L) = G(L)$, and in this case there is a Seifert matrix V associated to $S(L)$ with $V + V^t = G(L)$ (see §2, [GL]). This implies that $\sigma(\ell) = \sigma(H(L))$ and $\mathcal{N}(\ell) = \mathcal{N}(H(L))$. We define $\delta(A)$ to be the square free part of $|det(B)|$, where B is a nonsingular matrix associated to A. Then $\delta(\ell) = \delta(H(L))$ is also an invariant of links. Furthermore, the Minkowski unit $C_p(H(L))$ for $H(L)$ is an invariant of the link ℓ and we denote it by $C_p(\ell)$, for any prime integer p, including $p = \infty$[Le].

3. The modified Goeritz matrices of 2-periodic links

An (oriented) link $\ell = k_1 \cup \cdots \cup k_\mu$ of μ components in S^3 is called an *n-periodic (oriented) link* $(n > 1)$ if there is an orientation preserving homeomorphism $\phi : (S^3, \ell) \to (S^3, \ell)$ that satisfies the following properties:

1. $f = \{x \in S^3 | \phi(x) = x\}$ is homeomorphic to S^1,

2. ϕ is of order n, i.e., $\phi^n = id_{S^3} \neq \phi^k$ for $1 \leq k \leq n-1$,
3. $f \cap \ell = \emptyset$.

Let C_n be the group generated by the n-periodic homeomorphism ϕ for an n-periodic (oriented) link ℓ. Then the orbit space S^3/C_n is homeomorphic to a 3-sphere S^3. The quotient map $p : S^3 \to S^3/C_n$ is an n-fold cyclic covering map branched over $p(f) = f_*$ and $p(\ell) = \ell_*$ is also an (oriented) link in $S^3/C_n \cong S^3$, which is called the *factor link* of ℓ.

Theorem 4 [LP]. *Let ℓ be an oriented 2-periodic link in S^3 with the fixed point set $\bar{f}$ and let ℓ_* be the factor link of ℓ. Then there exist 2-periodic diagrams L and $L_* \cup \bar{F}_*$ of ℓ and $\ell_* \cup \bar{f}_*$ respectively satisfying the following:*

(1) $Lk(\ell, \bar{f}) \equiv 1 \pmod 2$.

$$S[H(L) \oplus \begin{pmatrix} I_a & O \\ O & -I_b \end{pmatrix} \oplus (2)]S^{-1} \approx \begin{pmatrix} H(L_*) & O \\ O & H(L_* \cup \bar{F}_*) \end{pmatrix}.$$

(2) $Lk(\ell, \bar{f}) \equiv 0 \pmod 2$.

Let $\ell \circ u$ denote the splittable 2-periodic link consisting of ℓ and the unknot u and let h^- denote the left handed Hopf link. Then

$$S[H(L \cup U) \oplus \begin{pmatrix} I_a & O \\ O & -I_{b+1} \end{pmatrix} \oplus (2)]S^{-1} \approx \begin{pmatrix} H(L_*) & O \\ O & H((L_* \cup \bar{F}_*) \sharp D^-) \end{pmatrix} \oplus (0),$$

where S is an invertible rational matrix, $L_* = \varphi_*(L)$, $L \cup U$ and D^- are diagrams of $\ell \circ u$ and h^- respectively, and $a - b + 1 = -Lk(\ell, \bar{f})$.

Corollary. *Let ℓ be a 2-periodic oriented link in S^3 and let ℓ_* be its factor link. Then*

(1) $\sigma(\ell) - Lk(\ell, \bar{f}) = \sigma(\ell_*) + \sigma(\ell_* \cup \bar{f}_*)$.

(2) $n(\ell) = n(\ell_*) + n(\ell_* \cup \bar{f}_*) - 1$,

where $\bar{f}_*$ denotes the knot f_* with an arbitrary orientation.

4. The 2-parallel version invariants.

Let B_n denote the braid group of n-strings. For a positive integer r, let $\phi_n^{(r)} : B_n \to B_{rn}$ be the group homomorphism defined by $\phi_n^{(r)}(\sigma_i)(1 \leq i \leq n-1) =$

$$\sigma(ri - r + 1, ri - 1)^{-r} \sigma(ri, ri + r - 1)\sigma(ri - 1, ri + r - 2) \cdots \sigma(ri - r + 1, ri),$$

where $\sigma(i,j) = \sigma_i \sigma_{i+1} \cdots \sigma_j$.

Let ℓ be an oriented link with μ-components and let $(\beta, n) \in B_n$ be a braid of n strings whose closure $\beta^\wedge$ is ambient isotopic to ℓ. The link $(\phi_n^{(r)}(b), rn)^\wedge$ is called the *r-parallel version of the link ℓ*, denoted by $\ell^{(r)}$.

Let (b_1, n_1) and (b_2, n_2) be two braids of n_1 and n_2 strings, respectivly. Then the links $(\phi_{n_1}^{(r)}(b_1), rn_1)^\wedge$ and $(\phi_{n_2}^{(r)}(b_2), rn_2)^\wedge$ are equivalent if (b_1, n_1) and (b_2, n_2) are Markov equivalent[Mu]. Let ρ be any invariant of oriented knots and links. For each oriented knot or link ℓ and a positive integer r, The invariant $\rho([\ell^{(r)}])$ is called the *r-parallel version of invariant ρ* of ℓ.

Theorem 5 [CLP]. *Let ℓ be a nonsplittable oriented link in S^3 and let $\beta = \sigma_{i_1}^{\tau_{i_1}}\sigma_{i_2}^{\tau_{i_2}}\cdots\sigma_{i_m}^{\tau_{i_m}} (\tau_{i_k} = \pm 1)$ be a braid word in B_n representing the link ℓ and $\bar{\beta}$ as above. Then the Goeritz matrix of the 2-parallel version $\ell^{(2)}$ of ℓ associated to $(\phi_n^{(2)}(\beta), 2n)^\wedge$ is given by the $(2m+1)\times(2m+1)$ integral matrix of the form:*

$$G(\beta) = \begin{pmatrix} W & B(\beta)^t & O \\ B(\beta) & O_{m\times m} & [r]^t \\ O & [r] & e_{n-1} \end{pmatrix},$$

where $B(\beta)$ is a matrix obtained from β, $[r] = (0\ 0\ \cdots\ 0\ \ -\tau(n-1,1)\ \ -\tau(n-1,2)\ \cdots\ -\tau(n-1,s_{n-1}))$, $e_{n-1} = \sum_{i=1}^{s_{n-1}} \tau(n-1,i)$, and W is an $m\times m$ symmetric integral matrix.

Corollary. *Let ℓ be a nonsplittable oriented link in S^3 and let $\beta = \sigma_{i_1}^{\tau_{i_1}}\sigma_{i_2}^{\tau_{i_2}} \cdots\sigma_{i_m}^{\tau_{i_m}} (\tau_{i_k} = \pm 1)$ be a braid word in B_n representing the link ℓ. Then the modified Goeritz matrix of the 2-parallel version $\ell^{(2)}$ of ℓ associated to $(\phi_n^{(2)}(\beta), 2n)^\wedge$ is given by*

$$H(\beta) = G(\beta) \oplus A(\beta),$$

where $G(\beta)$ is the Goeritz matrix of $\ell^{(2)}$ given in Theorem 3.1 and $A(\beta) = (diag(\tau_{i_1}, \tau_{i_2}, \cdots, \tau_{i_m}) \otimes I_2) \oplus (diag(-\tau_{i_1}, -\tau_{i_2}, \cdots, -\tau_{i_m}) \otimes I_2)$.

Theorem 6 [CLP]. *Let ℓ be a nonsplittable oriented link in S^3 of μ-components and let $\ell^{(2)}$ be the 2-parallel version of ℓ.*

$$0 \le |\sigma(\ell^{(2)})| \le \mathcal{N}(\ell^{(2)}) \le 2\mu.$$

Theorem 7 [CLP]. *Let ℓ be a nonsplittable link of μ-components in S^3 and let $\ell^{(2)}$ be the 2-parallel version of ℓ. If $n(\ell^{(2)}) = 2\mu$ and ℓ has a braid representative β such that $n(B(\beta)) = \mu - 1$. Then*

1. $\sigma(\ell^{(2)}) = 0$.
2. $\delta(\ell^{(2)}) = 1$.
3. $C_p(\ell^{(2)}) = 1$ *for any prime integer p, including $p = \infty$.*

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