Trends in Mathematics
Information Center for Mathematical Sciences
Volume 1, Number 1, September 1998, Pages 80–84

TOPOLOGICAL CHARACTERIZATIONS OF CERTAIN LIMIT POINTS

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ABSTRACT. A limit point p of a Möbius group acting on B^m is called a concentration point if for every sufficiently small connected open neighborhood of p, the set of translates contains a local basis for the topology of p. In this note, we investigate the general relations between several concentration properties.

1. INTRODUCTION

The action of a Fuchsian or Kleinian group on the sphere at infinity can be examined from several viewpoints, and the resulting interplay between topology, geometry, number theory, and analysis brings richness and beauty to the subject. The topological viewpoint provides the starting point for much of the theory, in that it gives the dichotomy between the region of discontinuity and the limit set. The region of discontinuity can be regarded as the portion of the sphere at infinity with trivial or nearly trivial local dynamics. In contrast, at points in the limit set the behavior is complicated and varied.

For a limit point p a well-known type of behavior is the property of being a conical limit point. This property is often defined geometrically by saying that there is a sequence of translates of the origin (where we regard the group , as acting on the Poincaré ball B^m) that limit to p and lie within a bounded hyperbolic distance of a geodesic ray ending at p. But it can also be described topologically in terms of the action of , on the sphere at infinity S^{m-1}. For example, one of several such characterizations is that there exist points $q \neq r$ in S^{m-1} and a sequence of distinct elements $\gamma_n \in$, such that $\gamma_n(p) \to q$ and $\gamma_n(x) \to r$ for every $x \in S^{m-1} - \{p\}$. For other topological characterizations of conical limit points, see [1, 3, 7].

Another topological aspect of the action of , on S^{m-1} is its concentration behavior. This refers to the action of , on the set of (open) neighborhoods of p in S^{m-1}. The following definitions appear in [1].

Definition: An open set U in S^{m-1} *can be concentrated* at p if for every neighborhood V of p, there exists an element $\gamma \in$, such that $p \in \gamma(U)$ and $\gamma(U) \subseteq V$. If in addition the element γ can always be selected so that $p \in \gamma(V)$, then one says that U can be concentrated *with control.*

Note that U can be concentrated at p if and only if the set of translates of U contains a local basis for the topology of S^{m-1} at p. Also, one can easily check from the definition that (1) there exists a neighborhood of p which can be concentrated with

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1991 *Mathematics Subject Classification.* 20H10, 30F35, 30F40,57M50.

Key words and phrases. Fuchsian group, Kleinian group, limit point, conical limit point, lamination, Schottky group, concentration point,controlled, weak, geodesic separation pointe.

control if and only if there is a connected neighborhood which can be concentrated with control (take the connected component of U that contains p, and require that $\gamma^{-1}(p) \in U \cap V$), and (2) if a neighborhood of p can be concentrated with control, then every smaller neighborhood can be concentrated with control.

Definition: The limit point p is called a *controlled concentration point* for , if it has a neighborhood which can be concentrated with control at p.

Concentration with control is studied in [1]. Analogously to conical limit points, p is a controlled concentration point if and only if there exist a point $r \neq p$ in S^{m-1} and a sequence γ_n of distinct elements of , so that $\gamma_n(p) \to p$ and $\gamma_n(x) \to r$ for all $x \in S^{m-1} - \{p\}$. In particular, every controlled concentration point is a conical limit point. However, examples are given in [1] of conical limit points of 2-generator Schottky groups which are not controlled concentration points. For groups of divergence type, controlled concentration points have full Patterson-Sullivan measure in the limit set. There is a direct connection between controlled concentration points and the dynamics of geodesics in the hyperbolic manifold $B^m/,$. Call a geodesic ray in $B^m/,$ *recurrent* if it is the image of a geodesic ray in B^m that ends at a controlled concentration point. In an appropriate metric, the space of recurrent geodesic rays in $B^m/,$ is a metric completion of the space of closed geodesics in $B^m/,$ (where both spaces are topologized as subspaces of the unit tangent bundle of $B^m/,$).

We turn now to weaker concentration properties. It is not difficult to show (see [9]) that every limit point p has a disconnected neighborhood that can be concentrated at p. So the weakest reasonable concept of concentration behavior is the following.

Definition: The limit point p is called a *weak concentration point* for , if there exists a connected open set that can be concentrated at p.

Weak concentration points are studied in [9]. It turns out that for a geometrically finite group, every limit point is a weak concentration point, and for any group, all but countably many limit points are weak concentration points. A more restrictive condition is that *every* sufficiently small connected neighborhood can be concentrated:

Definition: The limit point p is called a *concentration point* for , if every sufficiently small connected neighborhood of p can be concentrated at p.

A slightly weaker concept than concentration point for Fuchsian groups turns out to be important.

Definition: The limit point p is called a *geodesic separation point* for the Fuchsian group , if for every sufficiently small connected neighborhood U of p, either U or $S^1 - \overline{U}$ can be concentrated at p.

The name of this property derives from the fact that for a geodesic separation point p, if λ is any geodesic in B^2 whose endpoints separate p from the boundary of a small neighborhood of p, then for any neighborhood V of p there exists $\gamma \in$, so that the endpoints of $\gamma(\lambda)$ separate p from the boundary of V. Indeed, it is easily verified that this is equivalent to the condition in the definition; this simply uses the fact that every connected neighborhood of p (other than S^1 itself) is an interval, so corresponds to the unique geodesic in B^2 that runs between its endpoints.

Definition A geodesic λ is called a geodesic for , if both endpoints of λ are limit points of , . The limit point p is called a *Myrberg-Agard density point* for , if whenever μ is an oriented geodesic for , and α is a geodesic ray ending at p in $CH(\Lambda)$ (convex hull of Λ), there is a sequence of elements $\{\gamma_i\}$ such that $\{\gamma_i(\alpha)\}$ converge to μ in an oriented sense.

The related properties of Myrberg-Agard density points can be found in [4], [5] and [6].

The next three results contain the relation between controlled concentration points and geodesic laminations. The proof of those results appear in [8].

Lemma 1.1. *Let , be a torsionfree discrete group of Möbius transformations acting on the Poincaré disc B^m, and let $\pi\colon B^m \to B^m/,$ be the quotient map. Let $y_0 \in B^m$ and let p be a point in S^{m-1}. Let $\alpha\colon [0,\infty) \to B^m$ be the geodesic ray from y_0 to p, parameterized at unit speed. Suppose further that there exist numbers t_i, with $\alpha(t_i)$ limiting to p, so that in the tangent bundle $T(B^m/,)$, the images $d\pi(\alpha'(t_i))$ converge to $d\pi(\alpha'(0))$. Then p is a controlled concentration point for , .*

Theorem 1.2. *Let , be a Fuchsian group acting on the Poincaré disc B^2. Suppose there exists a geodesic ray in B^2 which ends at $p \in S^1$, which has no transverse crossing with any of its translates, and whose image in $B^2/,$ lies in a compact subset. Then p is a controlled concentration point for , .*

Corollary 1.3. *Let , be a torsionfree Fuchsian group, and let L be a compact geodesic lamination in $B^2/,$. Then the endpoints of the leaves of the preimage of L in B^2 are controlled concentration points for , .*

Note that the next proposition implies that for Fuchsian groups, concentration points are conical limit points. Whether this holds in higher dimensions is an open question.

Theorem 1.4. *Let , be a Fuchsian group. If , is finitely generated, then every limit point of , is either a parabolic fixed point or a geodesic separation point.*

2. Schottky groups and limit points

We will work with a 2-generator m-dimensional Schottky group , , although it will be apparent that the same phenomena occur for other examples (in particular, with more generators). The limit set of , is a Cantor set which can be understood quite explicitly using the sequence of crossings of a geodesic ray (ending at the limit point) with the translates of two fixed sides of a fundamental domain.

To define , , we work in the Poincaré unit disc B^m. Let a and a' be the geodesic hyperplanes in B^m which lie in the spheres in $\mathbf{R}^m$ with centers at the points (1.1,0,...,0) and (-1.1,0,...0), say. Similarly, let b and b' lie in the spheres with centers at the points (0,...,0,1.1) and (0,...,0,-1.1). Choose a, a', b, b' so that they are mutually disjoint. As the generators of , , select two orientation-preserving hyperbolic isometries: one carrying a to a' and one carrying b to b'. Fix one of the direction normal to a as the positive direction. It determines a positive normal direction for each translate of a. Similarly, we label b and its translates. A crossing of an oriented geodesic of geodesic ray in B^m with a translate of a or b will be called a *positive* crossing when it agrees with the selected direction; otherwise it will be called a *negative* crossing.

Suppose α is a geodesic ray in B^m, which does not lie in a translates of a and b. Then α crosses a sequence (finite or infinite, possibly of length 0) of translates of a and b. (When a geodesic ray starts in a translate, we count that intersection as a crossing.) To α, we associate a sequence $S(\alpha) = x_1x_2x_3$...of elements in the set $\{a, \overline{a}, b, \overline{b}\}$ in the following way. If the nth crossing of α with the union of the translates of a and b is a positive crossing with a translate of a, then $x_n = a$. If the nth crossing is a negative crossing with a translate of a, then $x_n = \overline{a}$. For crossings with translates of b, the elements b and $\overline{b}$ are assigned similarly. Note that $S(\alpha)$ is an infinite sequence if and only if α ends at a limit point of , , and note that, for each sequence $S = x_1x_2x_3...$ of elements of the set $\{a, \overline{a}, b, \overline{b}\}$ (with the property that for no n is x_nx_{n+1} in the set $\{a\overline{a}, \overline{a}a, b\overline{b}, \overline{b}b\}$), there exists a geodesic ray α with $S(\alpha) = S$.

Using these sequence, the controlled concentration points of , can be characterized. The following characterization appears in [1].

Proposition 2.1. *Let p be a limit point of , which is the endpoint of a geodesic ray α with $S(\alpha) = x_1x_2x_3\cdots$. Then p is a controlled concentration point for , if and only if $S(\alpha)$ has the following property. There exists N such that for all $n \geq N$, for all positive k, and for all M, there exists $m \geq M$ such that $x_{n+i} = x_{m+i}$ for all i with $0 \leq i \leq k$.*

The next two theorem work only for Schottky groups acting on B^2. Let , be a Schottky group with 2 generators as is described in the beginning of section 2 but we need to choose the geodesics in B^2 which lie in the spheres in $\mathbf{R}^2$ with centers at the points (1.1,0), (-1.1,0), (0,1.1) and (0,-1.1).

Denote by a_n a sequence of n a's, and by $\overline{a}_n$ a sequence of n $\overline{a}$'s. The following theorem gives examples of concentration points for a two generator Schottky group.

Theorem 2.2. *For each increasing sequence of positive integers $1 \leq i_1 < j_1 < i_2 < j_2 < i_3\cdots$, if p is a limit point which is the endpoint of a geodesic ray whose crossing is*

$$ba_{i_1}b\overline{a}_{j_1}ba_{i_2}b\overline{a}_{j_2}ba_{i_3}b\overline{a}_{j_3}\cdots$$

then p is a concentration point.

Theorem 2.3. *There are uncountably many limit points of $,_0$ which are not concentration points.*

Proposition 2.4. *There is an infinitely generated fuchsian group $,_1$, containing no parabolic elements, having uncountably many weak concentration points that are not conical limit points, and uncountably many conical limit points that are not geodesic separation points.*

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