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THE SEIBERG-WITTEN EQUATIONS AND SMOOTH 4-MANIFOLDS

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ABSTRACT. This is an expository article of recent developments in smooth 4-manifold theory. We present a brief explanation how the Seiberg-Witten theory is applied to smooth 4-manifold topology in the intersection forms. That is, we recast intersection forms of simply connected smooth 4-manifolds by using Seiberg-Witten theory.

1. INTRODUCTION

Since the inception of gauge theory, in the shape of the Yang-Mills instanton equations, Donaldson theory had played a central role in research of smooth 4-manifold topology. Even though Donaldson theory produced many remarkable results and uncovered some of the mysteries of smooth 4-manifold theory, most topologists felt it was difficult and complicated to understand the whole theory. In Fall 1994 Seiberg and Witten introduced a remarkable new type of gauge theory, called Seiberg-Witten theory ([W]). Despite that Seiberg-Witten theory is basically similar to Donalsdon theory, this new theory is much simpler and stronger than Donaldson theory in both results and techniques. For example, most results in smooth 4-manifold topology obtained from Donaldson theory are easily obtained by using Seiberg-Witten theory and some conjectures, notably Thom conjecture, are quickly settled down in this new theory.

In this article we give a brief explanation how the Seiberg-Witten theory is easily applied to get the same answers as we get in Donaldson theory, where proofs are long and difficult, in the intersection form problems of simply connected smooth 4-manifolds. It consists of two parts: Section 2 gives the basics of Seiberg-Witten theory – Seiberg-Witten equations, Seiberg-Witten invariants, and two fundamental theorems proved by Witten and Taubes, respectively. In section 3 we recast a proof of Donaldson's theorem "The intersection form of a simply connected, negative definite, smooth 4-manifold is standard" by using Seiberg-Witten theory. We also present some results on possible intersection forms of simply connected, spin, smooth 4-manifolds.

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2. Seiberg-Witten equations

In this section we briefly review the basics of Seiberg-Witten equations introduced by N. Seiberg and E. Witten (refer to [W],[KM],[M] for details).

Let X be an oriented, closed Riemannian 4-manifold, and let L be a characteristic line bundle on X, i.e. $c_1(L)$ is an integral lift of $w_2(X)$. This determines a $Spin^c$-structure on X which induces a unique complex spinor bundle $W \cong W^+ \oplus W^-$, where $W^\pm$ is the associated $U(2)$-bundles on X. Then $W^\pm \cong S^\pm \otimes L^{1/2}$ and $\det(W^\pm) \cong L$, where $S^\pm$ is a (locally defined) spinor bundle on X (refer to [LM] for details). For simplicity we assume that $H^2(X;\mathbf{Z})$ has no 2-torsion so that the set $Spin^c(X)$ of $Spin^c$-structures on X is identified with the set of characteristic line bundles on X.

Note that Clifford multiplication $c : T^*X \to \mathrm{Hom}(W^+, W^-)$ leads to an isomorphism

$$\rho : \Lambda^+ \otimes \mathbf{C} \longrightarrow sl(W^+)$$

taking Λ^+ to $su(W^+)$, and the Levi-Civita connection on TX together with a unitary connection A on L induces a connection $\nabla_A : , (W^+) \to , (T^*X \otimes W^+)$. This connection, followed by Clifford multiplication, induces a $Spin^c$-Dirac operator $D_A : , (W^+) \to , (W^-)$. The Seiberg-Witten equations ([W]) are the following pair of equations for a unitary connection A on L and a section Ψ of , (W^+) :

$$\text{(1)} \qquad \begin{cases} D_A\Psi &= 0 \\ \rho(F_A^+) &= -(\Psi \otimes \Psi^*)_0 \end{cases}$$

where F_A^+ is the self-dual part of the curvature of A and $(\Psi \otimes \Psi^*)_0$ is the trace-free part of $(\Psi \otimes \Psi^*)$ which is interpreted as an endomorphism of W^+.

The gauge group $\mathcal{G} := Aut(L) \cong Map(X, S^1)$ acts on the space $\mathcal{A}_X(L) \times , (W^+)$ by

$$g \cdot (A, \Psi) = (g \circ A \circ g^{-1}, g \cdot \Psi)$$

In particular, if $b_1(X) = 0$, then the gauge group $\mathcal{G}$ is homotopy equivalent to S^1 so that the quotient

$$\mathcal{B}_X^*(L) := \mathcal{A}_X(L) \times (, (W^+) - 0)/S^1$$

is homotopy equivalent to $\mathbf{C}P^\infty$. Since the set of solutions is invariant under the action, it induces an orbit space, called the (*Seiberg-Witten*) *moduli space*, denoted by $M_X(L)$, whose formal dimension is

$$\dim M_X(L) = \frac{1}{4}(c_1(L)^2 - 3\sigma(X) - 2e(X))$$

where $\sigma(X)$ is the signature of X and $e(X)$ is the Euler characteristic of X.

Definition A solution (A, Ψ) of the Seiberg-Witten equation (1) is called *irreducible* (*reducible*) if $\Psi \not\equiv 0$ ($\Psi \equiv 0$).

Note that if $b^+(X) > 0$ and $M_X(L) \neq \phi$, then for a generic metric on X the moduli space $M_X(L)$ contains no reducible solutions, so that it is a compact, smooth manifold of the given dimension. Furthermore the moduli space $M_X(L)$ is orientable and its orientation is determined by a choice of orientation on $\det(H^0(X;\mathbf{R}) \oplus H^1(X;\mathbf{R}) \oplus H^2_+(X;\mathbf{R}))$.

Definition The *Seiberg-Witten invariant* for a smooth 4-manifold X is a function $SW_X : Spin^c(X) \to \mathbf{Z}$ defined by

$$SW_X(L) = \begin{cases} 0 & \text{if } \dim M_X(L) < 0 \text{ or odd} \\ \sum_{(A,\Psi)\in M_X(L)} sign(A,\Psi) & \text{if } \dim M_X(L) = 0 \\ < \beta^{d_L}, [M_X(L)] > & \text{if } \dim M_X(L) := 2d_L > 0 \text{ and even} \end{cases}$$

Here $sign(A,\Psi)$ is ± 1 determined by an orientation on $M_X(L)$, and $\beta \in H^2(M_X(L);\mathbf{Z})$ is the first chern class of the $U(1)$-bundle

$$\widetilde{M_X}(L) = \{\text{solutions}(A,\Psi)\}/Aut^0(L) \longrightarrow M_X(L)$$

where $Aut^0(L)$ consists of gauge transformations which are the identity on the fiber of L over a fixed basepoint in X. For convenience, we denote the Seiberg-Witten invariant for X by $SW_X = \sum_L SW_X(L) \cdot e^L$, and a cohomology class $c_1(L) \in H^2(X;\mathbf{Z})$ is called a *Seiberg-Witten basic class* (for brevity, *SW-basic class*) for X if $SW_X(L) \neq 0$.

Note that if $b_2^+(X) > 1$, the Seiberg-Witten invariant $SW_X = \sum SW_X(L) \cdot e^L$ is a diffeomorphism invariant, i.e. SW_X does not depend on the choice of generic metric on X and generic perturbation of the Seiberg-Witten equation. Furthermore, only finitely many $Spin^c$-structures on X have a non-zero Seiberg-Witten invariant.

It turns out that the Seiberg-Witten theory has many powerful applications to smooth 4-manifolds. For example, it measures to some extent whether a given smooth 4-manifold is irreducible or not. That is, since the Seiberg-Witten invariant for a connected sum manifold $X = X_1 \sharp X_2$ with $b_2^+(X_i) > 0$ $(i=1,2)$ is identically zero, $SW_X \neq 0$ implies that X is irreducible unless X is homeomorphic to a blow-up manifold. Note that a smooth 4-manifold X is called *irreducible* if X is not a connected sum of other manifolds except for a homotopy sphere.

We close this section by mentioning two fundamental theorems of Seiberg-Witten theory. The first one is proved by Witten and the second one by Taubes.

Theorem 2.1 (Witten [W])**.** *Suppose X is a minimal algebraic surface of general type with $b_2^+ > 1$. Then the first chern class K_X of the canonical line bundle of X is the only (up to sign) SW-basic class and its SW-invariant is ± 1.*

Theorem 2.1 means that the canonical line bundle K_X of a minimal algebraic surface X of general type is diffeomorphic invariant, i.e. if $f : X \longrightarrow X'$ is an orientation-preserving diffeomorphism, then $f^*(K_{X'}) = \pm K_X$.

Theorem 2.2 (Taubes [T1],[T2])**.** *Let (X,ω) be a compact, oriented symplectic 4-manifold with $b_2^+(X) > 1$ and $\omega \wedge \omega$ giving the orientation. Then the Seiberg-Witten invariant of $K_X = -c_1(TX)$, which is the first chern class of the associated almost complex structure on X, is ± 1. Furthermore, any other SW-basic class κ of X satisfies*

$$|\kappa \cdot [\omega]| \leq K_X \cdot [\omega],$$

with equality if and only if $\kappa = \pm K_X$.

Theorem 2.2 implies many striking results on symplectic 4-manifolds, for example, a manifold not having SW-invariant ± 1 as well as a connected sum of two 4-manifolds with $b_2^+ > 1$ does not admit a symplectic structure.

3. Intersection forms of smooth 4-manifolds

The intersection form Q_X of a compact, simply connected 4-manifold X is an integral unimodular symmetric bilinear form

$$Q_X : H_2(X;\mathbf{Z}) \times H_2(X;\mathbf{Z}) \longrightarrow \mathbf{Z}$$

defined as an intersection number of two homology classes, or equivalently, as a cup product on $H^2(X;\mathbf{Z})$. The intersection form Q_X is a basic invariant of X determined by the oriented homotopy type of X ([Wh]). In 1982 Freedman showed that it is actually determined by the homeomorphism type of X ([Fr]).

Theorem 3.1 (Freedman [Fr]). *Every integral unimodular symmetric bilinear form is realized as the intersection form of one (two if it is odd form) compact, simply connected topological* 4*-manifold.*

But the story is strikingly different in smooth category. For example, some of integral quadratic forms are not realized as an intersection form of a compact, simply connected, smooth 4-manifold. The following is one of the main achievements in Donaldson theory whose proof is difficult and long ([FU]).

Theorem 3.2 (Donaldson [DK],[FU]). *If X is a compact, simply connected, smooth* 4*-manifold whose intersection form Q_X is negative definite, then Q_X is standard diagonalizable forms $n(-1)$.*

Sketch of Proof : ([F]) We prove this theorem in a single page using Seiberg-Witten theory. First suppose that Q_X is even, i.e. X is spin. We want to show that $H_2(X;\mathbf{Z})$. Consider the moduli space $M_X(\underline{\mathbb{C}})$ of Seiberg-Witten solutions corresponding to the trivial line bundle $L = \underline{\mathbb{C}}$, whose formal dimension is

$$\dim M_X(\underline{\mathbb{C}}) = \frac{c_1(\underline{\mathbb{C}})^2 - (2e(X) + 3\sigma(X))}{4} = -\frac{\sigma(X)}{4} - 1 = 2k - 1$$

where k is $\hat{A}(X)$-genus which is the index of the Dirac operator D on X. Note that the moduli space $M_X(\underline{\mathbb{C}})$ is non-empty because it contains a reducible solution $(A, 0)$. We investigate $M_X(\underline{\mathbb{C}})$ near this reducible soultion $(A, 0)$. Since the connections on the trivial line bundle $\underline{\mathbb{C}}$ can be identified with 1-forms on X, $M_X(\underline{\mathbb{C}})$ is exactly the zero set (modulo gauge) of a map

$$P : \Omega^1_X \times \Gamma(S^+) \longrightarrow \Omega^2_{+,X} \times \Gamma(S^-)$$

defined by $P(A, \Psi) = (F_A^+ - i(\Psi \otimes \Psi^*)_0, D_A\Psi)$. First linearize a map P and restrict it to $\ker d^* \times \Gamma(S^+)$:

$$\delta P = d^+ + D : \ker d^* \times \Gamma(S^+) \longrightarrow \Omega^2_{+,X} \times \Gamma(S^-)$$

Then the linearization δP is a Fredholm map, so that a neighborhood of $(A, 0)$ in $M_X(\underline{\mathbb{C}})$ is modelled on a quotient $L^{-1}(0)/\Gamma_{(A,0)}$, where $\Gamma_{(A,0)} \cong S^1$ is a stabilizer of $(A, 0)$ and

$$L : \ker\delta P \cong H^1(X;\mathbf{R}) \oplus \ker D \longrightarrow \operatorname{coker}\delta P \cong H^2_+(X;\mathbf{R}) \oplus \operatorname{coker} D$$

Since $H^1(X;\mathbf{R}) = H^2_+(X;\mathbf{R}) = 0$ and the index of D is k, L reduces to a map, called Kuranishi map

$$L : \mathbf{C}^{k+r} \longrightarrow \mathbf{C}^r$$

and a neighborhood of $(A, 0)$ in $M_X(\underline{\mathbb{C}})$ is modelled on $L^{-1}(0)/S^1$. Since L can be changed homotopically to a map L' which sends the last r-components in $\mathbf{C}^{k+r}$ to

$\mathbf{C}^r$ identically, a local model near $(A,0)$ in $M_X(\underline{\mathbb{C}})$ is $\mathbf{C}^k/S^1 \cong c(\mathbf{CP}^{k-1})$, a cone on $\mathbf{CP}^{k-1}$. Hence the complement $M'_X(\underline{\mathbb{C}}) := M_X(\underline{\mathbb{C}}) \setminus c(\mathbf{CP}^{k-1})$ is a compact, smooth $(2k-1)$-manifold with boundary $\mathbf{CP}^{k-1}$ because there is only one reducible solution $(A,0)$ in $M_X(\underline{\mathbb{C}})$.

Note that we have a S^1-fibration in page 3

$$\widetilde{M_X}(\underline{\mathbb{C}}) = \{\text{solutions}(A,\Psi)\}/Aut^0(\underline{\mathbb{C}}) \longrightarrow M_X(\underline{\mathbb{C}})$$

Restricting to the boundary $\mathbf{CP}^{k-1}$, it is the canonical S^1-fibration $S^{2k-1} \to \mathbf{CP}^{k-1}$, so a generator h of $H^2(\mathbf{CP}^{k-1};\mathbf{Z})$ is the restriction of the first chern class β of the fibration. Thus we have

$$1 =< h^{k-1}, [\mathbf{CP}^{k-1}] >=< \beta^{k-1}, \partial[M'_X(\underline{\mathbb{C}})] >= 0$$

which is a contradiction when $k > 0$. Hence $k = \hat{A}(X) = -\frac{\sigma(X)}{8} = 0$ implies that $H_2(X;\mathbf{Z}) = 0$.

In general case (i.e. if X is not spin), the same proof above works provided there exists a characteristic line bundle L on X which has a formal dimension

$$\dim M_X(L) = \frac{c_1(L)^2 - (2e(X) + 3\sigma(X))}{4} = \frac{c_1(L)^2 - \sigma(X)}{4} - 1 > 0$$

and we get the same contradiction. The existence of such a characteristic class follows from a number-theoretic result of N. Elkies ([E]):

> Let A be a negative definite $\mathbf{Z}$-inner product space. Then the shortest characteristic vectors have norm at most its rank(A), with equality if and only if A is standard.

This is the end of proof, which is an amazing application of Seiberg-Witten theory. □

Let us now turn to the case of indefinite forms. First, all odd indefinite forms are completely classified (up to isomorphic) as one of the following forms

$$\{m(1) \oplus n(-1) \mid m,\ n\ \in \mathbf{Z}^+\}$$

and all of these are realized as intersection forms of smooth 4-manifolds $m\mathbf{CP}^2 \sharp n\overline{\mathbf{CP}}^2$. Secondly, all even indefinite forms are algebraically classified as one of the following forms

$$\{2mE_8 \oplus nH \mid m,\ n\ \in \mathbf{Z}^+\}$$

but all of these are not realized as intersection forms of smooth 4-manifolds, where E_8 is the rank 8 negative definite intersection form obtained by the Dynkin diagram of E_8 and H is the intersection form of $S^2 \times S^2$. A big question related to intersection forms is "Which forms are realized as an intersection form of a smooth 4-manifold?" One of the most famous conjectures in smooth 4-manifold theory is the following, called 11/8-conjecture:

Conjecture 1. *If X is a simply connected, spin, smooth 4-manifold, then*

$$\frac{b_2(X)}{|\sigma(X)|} \geq \frac{11}{8}$$

equivalently, if we write $Q_X \cong 2mE_8 \oplus nH$, then $n \geq 3m$.

Though this conjecture is still unsettled, there has been some progress. For example, Donaldson and Kronheimer obtained partial results by using Donaldson theory and Seiberg-Witten theory, respectively. And Furuta recently announced a positive progress in this problem.

Theorem 3.3 (Donaldson [DK]). *If X is a simply connected, spin, smooth 4-manifold and $Q_X \cong 2mE_8 \oplus nH$ with $n \leq 2$, then $m = 0$.*

Theorem 3.4 (Kronheimer [F]). *If X is a simply connected, spin, smooth 4-manifold and $Q_X \cong 2mE_8 \oplus nH$ with $n \leq 8$, then $n \geq 3m$.*

Theorem 3.5 (Furuta [Fur]). *Suppose X is a simply connected, spin, smooth 4-manifold with $Q_X \cong 2mE_8 \oplus nH, n \neq 0$. Then $n \geq 2m + 1$.*

References

[D] S. Donaldson, The Seiberg-Witten equations and 4-manifold topology, Bull. Amer. Math. Soc. **33** no **1** (1996), 45-70

[DK] S. Donaldson and P. Kronheimer, The geometry of four-manifolds, Oxford, 1990

[E] N. Elkies, A characterization of the $\mathbf{Z}^n$-lattice, Mathematical Research Letters, to appear

[Fr] M. Freedman, The topology of four-dimensional manifolds, J. Diff. Geo. **17** (1982), 357-454

[F] R. Fintushel, New directions in 4-manifold theory, Lecture at Michigan State University, 1995

[FU] D. Freed and K. Uhlenbeck, Instantons and four-manifolds, MSRI publications, 1984

[Fur] M. Furuta, Monopole equation and the 11/8 conjecture, preprint

[LM] H. Lawson and M-L. Michelson, Spin geometry, Princeton University Press, Princeton, 1989

[M] J. Morgan, The Seiberg-Witten equations and applications to the topology of smooth four-manifolds, Princeton University Press, Princeton, 1996

[KM] P. Kronheimer and T. Mrowka, The genus of embedded surfaces in the projective plane, Mathematical Research Letters **1** (1994), 797-808

[T1] C. Taubes, The Seiberg-Witten invariants and symplectic forms, Mathematical Research Letters **1** (1994), 809-822

[T2] C. Taubes, More constraints on symplectic forms from Seiberg-Witten equations, Mathematical Research Letters **2** (1995), 9-14

[W] E. Witten, Monopoles and four-manifolds, Mathematical Research Letters **1** (1994), 769-796

[Wh] J.H.C. Whitehead, On simply connected 4-dimensional polyhedra, Comm. Math. Helvetici **45** (1949), 48-92

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