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GROUP ACTIONS ON THE 3-DIMENSIONAL NILMANIFOLD

JOONKOOK SHIN

ABSTRACT. We study free actions of finite abelian groups on the 3-dimensional nilmanifold, up to topological conjugacy. By the works of Bieberbach and Waldhausen, this classification problem is reduced to classifying all normal nilpotent subgroups of almost Bieberbach groups of finite index, up to affine conjugacy.

1. INTRODUCTION

The general question of classifying finite group actions on a closed 3-manifold is very hard. However, the actions on a 3-dimensional torus can be understood easily by the works of Bieberbach and Waldhausen([5, 8]). We shall study only free actions of finite abelian groups G on the 3-dimensional nilmanifold.

A group N is p-step nilpotent if $N^{(p)} = 1$, where $N^{(1)} = [N, N]$, the commutator subgroup of N, and $N^{(i+1)} = [N, N^{(i)}]$. Nil denotes the 3-dimensional Heisenberg group; i.e. Nil consists of all 3×3 real upper triangular matrices with diagonal entries 1. Thus Nil is a simply connected, nilpotent Lie group, and it fits an exact sequence

$$1 \to \mathbb{R} \to \mathrm{Nil} \to \mathbb{R}^2 \to 1$$

where $\mathbb{R} = \mathcal{Z}(\mathrm{Nil})$, the center of Nil. Hence Nil has the structure of a line bundle over $\mathbb{R}^2$. We take a left invariant metric coming from the orthonormal basis

$$\left\{ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

for the Lie algebra of Nil. This is, what is called, the Nil-geometry and its isometry group is $\mathrm{Isom(Nil)} = \mathrm{Nil} \rtimes O(2)$ [10, 11]. All isometries of Nil preserve orientation and the bundle structure.

We say that a closed 3-dimensional manifold M has a Nil–geometry if there is a subgroup π of Isom(Nil) so that π acts properly discontinuously and freely with compact quotient $M = \mathrm{Nil}/\pi$. The simplest such a manifold is the quotient of Nil by the integral lattice of Nil. The integral lattice means the subgroup of Nil consisting of matrices with integer entries.

Let $\mathrm{Aff}(n) = \mathbb{R}^n \rtimes \mathrm{GL}(n, \mathbb{R})$ denote the group of affine motions of $\mathbb{R}^n$, and let $E(n) = \mathbb{R}^n \rtimes \mathrm{O}(n)$ be the group of the rigid motions of $\mathbb{R}^n$. The group operation is

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given by $(a, A)(b, B) = (a + Ab, AB)$ and $\mathrm{Aff}(n)$ acts on $\mathbb{R}^n$ by $(a, A)x = a + Ax$ for (a, A), $(b, B) \in \mathrm{Aff}(n)$ and $x \in \mathbb{R}^n$. A discrete uniform subgroup of $E(n)$ is called an n-dimensional crystallographic group. Therefore, torsion-free crystallographic groups, called Bieberbach groups, are exactly the fundamental groups of compact flat Riemannian manifolds. A crystallographic group is characterized algebraically as follows [13].

Theorem 1.1. *An abstract group E is isomorphic to an n-dimensional crystallographic group if and only if E has a normal free abelian subgroup , of rank n and finite index which is maximal abelian in E.*

It follows from theorem 1.1 that there is an exact sequence

$$1 \to \,, \to E \to \Phi \to 1$$

where $, = E \cap \mathbb{R}^n$ is a normal free abelian subgroup of rank n which is maximal abelian in E. The finite group $\Phi = E/, \subset O(2)$ is called the linear holonomy group of E. For more details, see [1].

Let L be a connected and simply connected nilpotent Lie group. Then $\mathrm{Aff}(L) = L \rtimes \mathrm{Aut}(L)$ is called the affine group of L, where the group operation is given by $(g, \alpha)(h, \beta) = (g \cdot \alpha(h), \alpha\beta)$ and $\mathrm{Aff}(L)$ acts on L by $(g, \alpha)z = g \cdot \alpha(z)$ for $(g, \alpha), (h, \beta) \in \mathrm{Aff}(L)$ and $z \in L$. Let K be any maximal compact subgroup of $\mathrm{Aut}(L)$. Then a discrete uniform subgroup E of $L \rtimes K$ is called an *almost crystallographic group*. When E is a torsion-free, it is called an *almost Bieberbach group* and the coset space $E\backslash L$ is an infra-nilmanifold. (In case $E \subset L$, $E\backslash L$ is called a nilmanifold.) If L is abelian ($\cong \mathbb{R}^n$ for some n), this terminology reduces to a *crystallographic group*, a *Bieberbach group* and a *flat Riemannian manifold* respectively.

The maximal compact subgroup K can be chosen so that $L \rtimes K$ equals $\mathrm{Isom}(L)$. Therefore, almost Bieberbach groups are exactly the fundamental groups of compact infra-nilmanifolds. Consequently, a closed 3–dimensional manifold has a Nil–geometry if and only if it is an infra–nilmanifold. It is well known that infra-nilmanifolds are determined completely (up to affine diffeomorphism) by their fundamental group E.

Let G be a finite group acting freely on a nilmanifold $\mathcal{N}$. Then clearly, $M = \mathcal{N}/G$ is a topological manifold, and $, = \pi_1(\mathcal{N}/G)$ is an abstract Bieberbach group. Let N be the subgroup of , corresponding to $\pi_1(\mathcal{N})$. Let $,'$ be an embedding of , into $\mathrm{Aff}(L)$ as a cocompact subgroup, and let N' be the image of N. Then the quotient group $G' = ,'/N'$ acts freely on the nilmanifold $\mathcal{N}' = L/N'$. Moreover, $M' = \mathcal{N}'/G'$ is an infra-nilmanifold. Thus, a finite free topological action $(G, \mathcal{N})$ gives rise to an isometric action $(G', \mathcal{N}')$ on a nilmanifold. Clearly, $\mathcal{N}/G$ and $\mathcal{N}'/G'$ are sufficiently large, see [4, Proposition 2]. By works of Waldhausen and Heil ([3, Theorem A]), M is homeomorphic to M'.

Definition 1.2. Let groups G_i act on manifolds M_i, for $i = 1, 2$. The action (G_1, M_1) is *topologically conjugate* to (G_2, M_2) if there exists an isomorphism $\theta\colon G_1 \to G_2$ and a homeomorphism $h\colon M_1 \to M_2$ such that $h(g \cdot x) = \theta(g) \cdot h(x)$ for all $x \in M_1$ and all $g \in G_1$. When $G_1 = G_2$ and $M_1 = M_2$, topologically conjugate is the same as *weakly equivariant*.

For $\mathcal{N}/G$ and $\mathcal{N}'/G'$ being homeomorphic implies that the two actions $(G, \mathcal{N})$ and $(G', \mathcal{N}')$ are topologically conjugate. Consequently, a free finite action $(G, \mathcal{N})$ gives rise to a topologically conjugate isometric action $(G', \mathcal{N}')$ on a nilmanifold $\mathcal{N}'$. Such a pair $(G', \mathcal{N}')$ is not unique. However, by the following theorem which has been obtained by Lee and Raymond ([7]), all the others are topologically conjugate.

Theorem 1.3. *Let $\phi : E \to E'$ be an isomorphism between two almost crystallographic groups of L. Then ϕ is conjugation by an element of* $\mathrm{Aff}(L)$.

Consequenly, to classify all free actions by finite groups on a nilmanifold, it is enough to classify only free isometric actions by finite groups on a nilmanifold. Lee proved the following theorem([6]).

Theorem 1.4. *Let $1 \to N \to E \to F \to 1$ be any extension of a finitely generated, torsion-free nilpotent group N by a finite group F. Then there exists a finite characteristic subgroup Z of E such that E/Z is an almost crystallographic group. In fact, Z is the subgroup consisting of all torsion elements of $C_E(N)$, the centralizer of N in E.*

Therefore, a finitely generated group E is an almost crystallographic group if and only if it has a torsion-free maximal normal nilpotent subgroup of finite index.

2. Criteria for conjugacy

In this section, we develop a technique for finding and classifying all possible finite group actions on the 3-dimensional nilmanifold. The problem will be reduced to a purely group-theoretic one.

Throughout this paper, we shall denote the Heisenberg group Nil simply by $\mathcal{H}$.

Let , be any lattice of $\mathcal{H}$. Then $\mathbb{Z} = , \cap \mathcal{Z}(\mathcal{H})$ and $, /, \cap \mathcal{Z}(\mathcal{H})$ are lattices of $\mathcal{Z}(\mathcal{H})$ and $\mathcal{H}/\mathcal{Z}(\mathcal{H})$,respectively. Therefore, the lattice , is an extension of $\mathbb{Z}$ by $\mathbb{Z}^2$, that is,

$$1 \to \mathbb{Z} \to \, , \to \mathbb{Z}^2 \to 1$$

We take e_1, e_2 and e_3 in , for some $k(\neq 0) \in \mathbb{Z}$, where

$$e_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ e_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \ e_3 = \begin{bmatrix} 1 & 0 & 1/k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus

$$, = \langle e_1, e_2, e_3 \mid [e_1, e_3] = [e_2, e_3] = 1, \ [e_1, e_2] = e_3^k \rangle.$$

It is easy to see that every lattice of $\mathcal{H}$ is isomorphic to , for some $k > 0$. ¿From now on, we denote the lattice determined as above by $,_k$. A group is *virtually p-step nilpotent* if it contains a normal subgroup which is p-step nilpotent and of finite index. Proposition 2.1 gives a characterization of an almost Bieberbach group.

Proposition 2.1. *An abstract group π is the fundamental group of a 3-dimensional infra-nilmanifold if and only if π is a torsion-free, virtually nilpotent group of rank 3. Consequently, such a π contains $,_k$ for some $k > 0$ as a normal subgroup of finite index.*

Proof. Suppose that M is a 3-dimensional infra-nilmanifold Then M is finitely covered by a nilmanifold $,\backslash\mathcal{H}$, where , is a lattice of $\mathcal{H}$. Then necessarily, $, = ,_k$

for some $k > 0$, and hence,

$$1 \to \, , _k \to \pi_1(M) \to \Phi \to 1$$

is an extension of $, _k$ by a finite group Φ. Thus $\pi_1(M)$ contains $, _k$ as a nilpotent subgroup of rank 3. Indeed, $, _k$ is the unique maximal normal nilpotent subgroup of $\pi_1(M)$.

Conversely, π fits an exact sequence $1 \to N \to \pi \to F \to 1$ where N is a finitely generated, torsion-free nilpotent group and F is a finite group. Then clearly $N = \, , _k$ for some k. Thus Theorem 1.4 completes the proof.

In fact, the fundamental group of an infra-nilmanifold determines the space completely as in the case of a compact flat Riemannian manifold.

Theorem 2.2. *Let M_1, M_2 be infra-nilmanifolds. Then the following conditions are equivalent.*

(1) *M_1 is affinely diffeomorphic to M_2.*
(2) *M_1 is homeomorphic to M_2.*
(3) *The fundamental group $\pi_1(M_1)$ is isomorphic to $\pi_1(M_2)$.*

Proof. The statement (3) is equivalent to M_1 being homotopy equivalent to M_2, since both are aspherical manifolds. Now, homotopy equivalent infra-nilmanifolds are affinely diffeomorphic ([7]), so (3) implies (1).

Let $M = \mathcal{H}/\pi$ be a 3-dimensional infra-nilmanifold where π is a subgroup of $\mathrm{Isom}(\mathcal{H}) = \mathcal{H} \rtimes O(2)$. Then there is a diffeomorphism f between $\mathcal{H}$ and $\mathbb{R}^3$, and an isomorphism φ between π and π' where π' is a subgroup of $\mathrm{Aff}(3) = \mathbb{R}^3 \rtimes \mathrm{GL}(3,\mathbb{R})$ such that $(\pi, \mathcal{H})$ and $(\pi', \mathbb{R}^3)$ are weakly equivariant. Therefore, an infra-nilmanifold $M = \mathcal{H}/\pi$ is diffeomorphic to an affine manifold $M' = \mathbb{R}^3/\pi'$.

Definition 2.3. Let $, \subset \mathrm{Aff}(\mathcal{H})$ be an almost Bieberbach group, and let N_1, N_2 be subgroups of , . We say that $(N_1, ,)$ is *affinely conjugate* to $(N_2, ,)$ if there exists an element $(t,T) \in \mathrm{Aff}(\mathcal{H})$ such that $(t,T) \, , (t,T)^{-1} = \, ,$ and $(t,T)N_1(t,T)^{-1} = N_2$.

Let $(G, \mathcal{N})$ be a free affine action of a finite abelian group G on the nilmanifold $\mathcal{N}$. Then $\mathcal{N}/G$ is an infra-nilmanifold. Let $, = \pi_1(\mathcal{N}/G)$, and $N = \pi_1(\mathcal{N})$. Then , is an almost Bieberbach group. In fact, since the covering projection $\mathcal{N} \to \mathcal{N}/G$ is regular, N is a normal subgroup of , . Since the pure translations in , , $\Delta = \, , \cap \mathcal{H}$, is the unique maximal normal nilpotent subgroup of , , the normal subgroup N must be in Δ.

Our classification problem of free abelian group actions $(G, \mathcal{N})$ with $\pi_1(\mathcal{N}/G) \cong$, can be solved by two steps:

(I) Find all normal nilpotent subgroups N of , of finite index and classify $(N, ,)$ up to affine conjugacy.
(II) Realize the finite abelian group $G = \, , /N$ as an action on the nilmanifold $\mathcal{H}/N$.

For (I), we need the following. Let $, \subset \mathrm{Aff}(\mathcal{H}) = \mathcal{H} \rtimes \mathrm{Aut}(\mathcal{H})$ be an almost Bieberbach group; and let

$$t_1 = (e_1, I), \qquad t_2 = (e_2, I), \qquad t_3 = (e_3, I),$$

where I is the identity matrix in $\mathrm{Aut}(\mathcal{H}) = \mathbb{R}^2 \rtimes \mathrm{GL}(2,\mathbb{R})$.

Then we have the following Lemma.

Lemma 2.4. *Any normal nilpotent subgroup N of , has an ordered set of generators of the form*

$$\langle t_1^{d_1} t_2^{m} t_3^{n_1},\ t_2^{d_2} t_3^{n_2},\ t_3^{d_3}\rangle.$$

Let us denote the normalizer of , by $N_{\mathrm{Aff}(\mathcal{H})}(,\ )$. The maximal normal nilpotent subgroup $\mathcal{Z}$ of , is characteristic (i.e., invariant under any automorphism of ,). Under our representation of , into $\mathrm{Aff}(\mathcal{H})$, the subgroup , is a lattice of $\mathcal{H}$. Therefore, matrix parts of elements of $N_{\mathrm{Aff}(\mathcal{H})}(,\ )$ are integral.

To make the exposition easier, we introduce some more notations. Let N_1, N_2 be nilpotent normal subgroups of , ; $\mathcal{B}_1$, $\mathcal{B}_2$ be bases for N_1, N_2, respectively. If there is $Y \in \mathrm{Aut}(,\ ) \subset \mathrm{Aut}(\mathcal{H})$ such that $[\mathcal{B}_1]Y = [\mathcal{B}_2]$, then we say $[\mathcal{B}_1] \underset{C}{\sim} [\mathcal{B}_2]$. Similarly, if there exists $(t,T) \in N_{\mathrm{Aff}(\mathcal{H})}(,\ )$ so that $[\mathcal{B}_2] = t \cdot T([\mathcal{B}_1]) \cdot t^{-1}$, then we say $[\mathcal{B}_1] \underset{R}{\sim} [\mathcal{B}_2]$. Note that $\underset{C}{\sim}$ is the right action of $\mathrm{Aut}(N)$ on a fixed basis so that it does not change , and its normal subgroup. It is an operation that picks a new set of generators. Therefore, if $[\mathcal{B}_1] \underset{C}{\sim} [\mathcal{B}_2]$, then $N_1 = N_2$. On the other hand, $\underset{R}{\sim}$ is the row operation on the matrix leaving , invariant. If (t,T) is in the normalizer of , , then it gives a new representation of , . Moreover, even if $[\mathcal{B}_1] \underset{R}{\sim} [\mathcal{B}_2]$, N_1 and N_2 will generally be different subgroups of , .

The following proposition is a working criterion for affine conjugacy. All calculations will be done by this method.

Proposition 2.5. *Let N_1, N_2 be normal nilpotent subgroups of an almost Bieberbach group , . Then $(N_1, ,\ )$ is affinely conjugate to $(N_2, ,\ )$ if and only if for any ordered set of generators $\mathcal{B}_1, \mathcal{B}_2$ for N_1, N_2, respectively, there exist $(t,T) \in N_{\mathrm{Aff}(\mathcal{H})}(,\ )$ and $Y \in GL(2,\mathbb{Z})$ such that*

$$[\mathcal{B}_2]Y = t \cdot T([\mathcal{B}_1]) \cdot t^{-1}.$$

3. Free actions of finite abelian groups **G** on the nilmanifold

It is well known that all 3-dimensional infra-nilmanifolds are Seifert manifolds. A classification of the 3-dimensional Seifert manifolds with solvable fundamental group (amongst them infra-nilmanifolds) is found in Orlik's book ([9, Theorem 1., p.142]). Assume M is a 3-dimensional infra-nilmanifold. Then M has a Seifert bundle structure; namely, M is a circle bundle over a 2-dimensional orbifold with singularities. Recently it is known ([2, Proposition 6.1.]) that there are only 15 kinds of distinct closed 3-dimensional manifolds M with a Nil-geometry up to Seifert local invariant.

In this section, we shall deal with only one out of 15 distinct almost Bieberbach groups up to Seifert local invariant. The other cases can be done similarly and will be announced([12]). ¿From now on, let us denote , by

$$,\ = \langle t_1, t_2, t_3, \alpha \mid [t_1, t_2] = t_3^{2k},\ \alpha^2 = t_3,\ \alpha t_1 \alpha^{-1} = t_1^{-1},\ \alpha t_2 \alpha^{-1} = t_2^{-1}\rangle,$$

where $t_1 = (e_1, I)\,, t_2 = (e_2, I)\,, t_3^{2k} = (e_3, I)$, and

$$\alpha = \left(\begin{bmatrix} 1 & 0 & \frac{1}{4k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right).$$

Here I is the identity matrix in $\mathrm{Aut}(\mathcal{H}) = \mathbb{R}^2 \rtimes \mathrm{GL}(2,\mathbb{R})$. Note that $[\,,\,,\,] = \langle t_1^{-2}, t_2^{-2}, t_3^{2k}\rangle$, its holonomy group is $\mathbb{Z}_2$ and the fist homology group is $H_1(\,,\,;\mathbb{Z}) = \mathbb{Z}_{4k} \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Recall that for integers p and q,

$$N_{\mathrm{Aff}(\mathcal{H})}(\,,\,) = \left\{ t = \begin{bmatrix} 1 & \frac{p}{2} & * \\ 0 & 1 & \frac{q}{2} \\ 0 & 0 & 1 \end{bmatrix}, T \in \mathrm{GL}(2,\mathbb{Z}) \right\}.$$

We shall study free actions of finite abelian groups G on the nilmanifold $\mathcal{N}$ which yield an infra-nilmanifold homeomorphic to $\mathcal{H}/\,,$.

Theorem 3.1. *The following table gives a complete list of all free actions (up to topological conjugacy) of finite abelian groups G on $\mathcal{N}$ which yield an orbit manifold homeomorphic to $\mathcal{H}/\,,$.*

Group G	*Conjugacy classes of normal free subgroups*
$\mathbb{Z}_{4k}$	$K = \langle \alpha^{4k}, t_1, t_2\rangle$
$\mathbb{Z}_{8k} \times \mathbb{Z}_2$	$N_1 = \langle \alpha^{8k}, t_1, t_2^2\rangle$
	$N_2 = \langle \alpha^{8k}, t_1 t_2 t_3^k, t_2^2\rangle$
$\mathbb{Z}_{16k} \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$L = \langle \alpha^{16k}, t_1^2, t_2^2\rangle$

The action of $\,/K = \mathbb{Z}_{4k}$ on the nilmanifold $\mathcal{H}/K$ is given by $\langle h\rangle$:

$$h(x,y,z) = (-x,\, -y,\, z + \frac{1}{4k}).$$

The action of $\,/N_1 = \mathbb{Z}_{8k} \times \mathbb{Z}_2$ on the nilmanifold $\mathcal{H}/N_1$ is given by

$$\langle f, g\rangle : f(x,y,z) = (-x,\, -y,\, z + \frac{1}{8k}), \qquad g(x,y,z) = (x, y + \frac{1}{2}, z).$$

The action of $\,/L = \mathbb{Z}_{16k} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ on the nilmanifold $\mathcal{H}/L$ is given by $\langle \phi, \xi, \eta\rangle$:

$$\phi(x,y,z) = (-x,\, -y,\, z + \frac{1}{16k}),\ \xi(x,y,z) = (x + \frac{1}{2}, y, z + \frac{y}{2}),\ \eta = g.$$

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Chungnam National University, Taejeon 305–764, Korea
E-mail address: jkshin@math.chungnam.ac.kr