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DIFFERENTIAL OPERATORS AND AUTOMORPHIC FORMS

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Abstract. Differential operators play important roles in the theory of automorphic forms. In this lecture, we introduce how to construct automorphic forms using differential operators and study differential equations satisfied by automorphic forms. The recent results related to Jacobi forms, Siegel modular forms as well as Hilbert modular forms have been discussed.

1. Constructions of the Automorphic forms

Classically, there are many interesting connections between differential operators and the theory of automorphic forms and many interesting results have been explored.

For instance, it has been known for some time how to obtain an automorphic form from the derivatives of N automorphic forms. The case $N = 1$ has already been studied in detail by R. Rankin in 1956 [18].
Let f be an automorphic forms of weight k on , where , is a discrete subgroup of $SL(2, \mathbb{R})$ acting on the complex upper half plane $\mathcal{H}$. A holomorphic function $f : \mathcal{H} \to \mathbb{C}$ is called an automorphic form of weight k on , if it satisfies

$$(c\tau + d)^{-k} f(\gamma\tau) = f(\tau), \forall \gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix} \right) \in ,$$

with a bounded condition at each cusp
Rankin has found all polynomials(with complex coefficients) in f and its derivatives which are automorphic forms for , . His results are expressed as polynomials in f and certain auxiliary functions $\Phi_m(f), m = 2, 3...$ defined by

$$\Phi_m(f) = (-1)^m \begin{vmatrix} \frac{f_1}{1!\Gamma(k+1)}, & \frac{2f_2}{2!\Gamma(k+2)}, & .. & .. & \frac{mf_m}{m!\Gamma(k+m)}, \\ \frac{f}{\Gamma(k)}, & \frac{f_1}{1!\Gamma(k+1)}, & \frac{f_2}{2!\Gamma(k+2)}, & \cdots & \frac{f_{m-1}}{(m-1)!\Gamma(k+m-1)}, \\ 0, & \frac{f}{\Gamma(k)}, & \frac{f_1}{1!\Gamma(k+1)}, & \cdots & \frac{f_{m-2}}{(m-2)!\Gamma(k+m-2)}, \\ .... & & & & \\ 0 & ... & 0 & \frac{f}{\Gamma(k)}, & \frac{f_1}{1!\Gamma(k+1)} \end{vmatrix}$$

where $f_n = \frac{d^n f}{dz^n}$ and , (s) is the Euler gamma function. Later, he extended his result include polynomials in several distinct automorphic forms and their derivatives[19] (Recently, all polynomial differential operators which map a Jacobi form to a Jacobi form have been also determined[8]).
When $N = 2$, as a special case, H. Cohen has constructed certain covariant bilinear operators which he used to obtain modular forms of half integral weight

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with interesting Fourier coefficients [10]. Later, these operators were called Rankin-Cohen operators by D. Zagier who studied their algebraic relations [11, 24]; let $M_m(,)$ be the space of automorphic forms of weight m on , . A map, for each nonnegative integer ν, $[\,,\,]_\nu : M_k(,) \times M_\ell(,) \to M_{k+\ell+2\nu}(,)$ defined by

(1.1)

$$[f,\,,g]_\nu(\tau) = \sum_{r+s=\nu} (-1)^r \begin{pmatrix} \nu+k-1 \\ s \end{pmatrix} \begin{pmatrix} \nu+\ell-1 \\ r \end{pmatrix} D^r(f(\tau))D^s(g(\tau)), D = \frac{d}{d\tau},$$

is called the νth Rankin-Cohen bracket(operator). It was shown that the space of bilinear holomorphic differential operators on the space of automorphic forms is one dimensional, *i.e*, the bilinear operator on the space is of the form in (1.1) up to constant[24]. One can also study multi linear differential operators.

Recently, more generally, the Rankin-Cohen type bilinear differential operators and their applications have been studied on the spaces of Jacobi forms, Siegel modular forms as well as Hilbert modular forms[10, 24, 2, 3, 5, 12, 6]. For instance, the following states the existence of the Rankin-Cohen brackets on the space of Jacobi forms of weight k and index m on , (see [13] for the definition of Jacobi forms);

let $J_{k,m}(,)$ be a space of Jacobi forms of weight k and index m on , . A map, for any $X \in \mathbb{C}$ and each nonnegative integer ν,

$$[[\,,\,]]_{X,\nu} : J_{k,m}(,) \times J_{k',m'}(,) \to J_{k+k',m+m'}(,),$$

defined by

$$[[f,f']]_{X,\nu} = \sum_{r+s+p=\lfloor\nu/2\rfloor,\, i+j=\nu-2\lfloor\nu/2\rfloor} C_{r,s,p}(k,k')D_{r,s,i,j}(m,m',X)L^p_{m+m'}\left(L^r_m(\partial^i_z f)L^s_{m'}(\partial^j_z f')\right)$$

where

$$\begin{aligned} D_{r,s,i,j}(m,m',X) &= m^j(-m')^i(1+mX)^s(1-m'X)^r\,, \\ C_{r,s,p}(k,k') &= \frac{(\alpha+r+s+p)_{s+p}}{r!}\cdot\frac{(\beta+r+s+p)_{r+p}}{s!}\cdot\frac{(-(\gamma+r+s+p))_{r+s}}{p!} \\ &\quad (\alpha = k-3/2,\ \beta = k'-3/2,\ \gamma = k+k'-3/2+(\nu-2\lfloor\nu/2\rfloor))\,, \end{aligned}$$

where $(x)_m = \prod_{0\le i\le m-1}(x-i)$, $\lfloor x\rfloor$ denotes the largest integer $\le x$, and, where $L_m(f) = (8\pi i m\frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial z^2})f$ for f a Jacobi form of index m.

For fixed ν and k, m and k', m' large enough the operators $[\cdot,\cdot]_{X,\nu}$ ($X \in \mathbb{C}$) span a vector space of dimension $\lfloor\frac{\nu}{2}\rfloor + 1$ This shows that the space of such bilinear holomorphic differential operators is, in general, at least $\lfloor\frac{\nu}{2}\rfloor + 1$ dimensional. A result of Böcherer [1], obtained by using Maaß operators, shows that this dimension actually equals $\lfloor\frac{\nu}{2}\rfloor + 1$ in general.

The Rankin-Cohen brackets can be used to get the arithmetic informations of the automorphic forms. For instance, one can study an interesting L-series by computing the Petersson inner product of an automorphic form f against the Rankin-Cohen brackets on the automorphic forms. The arithmetic applications of Rankin-Cohen brackets on the space of Jacobi forms have been already explored[9].

2. Differential Equations

In 1848 Jacobi published a third order non-linear differential equation satisfied by the theta "Nullwerte"[17]. About 50 years later, Hurwitz proved that every automorphic form or function associated with a horocyclic group satisfies a non-linear algebraic differential equation of order at most three [16, 21](Of course, it doesn't mean that it is easy to find such a differential equation). Much later, in 1972, Resnikoff proved a partial converse to the theorem of Hurwitz[20]; *a non constant holomorphic automorphic form attached to a horocyclic group acting on the upper half plane cannot satisfy an algebraic differential equation of order less than* 3(By an algebraic differential equation we mean one of the form

$$0=\sum a(n_0,...,n_k)f^{n_0}f_1^{n_1}...f_k^{n_k}, \frac{d^jf}{d\tau^j}\equiv f_j, a(n)\in\mathbb{C}).$$

(One should remind that although a third order differential equation for modular forms certainly exists, it is by no means that it will be the simplest or most convenient with which to work. In fact, an explicit non homogeneous differential equation of order 4 for Weierstrass's discriminant $\Delta(\tau)$ had been found by Van Der Pol(see [20] or [22]);

$$\frac{\Delta''''}{\Delta}-5\frac{\Delta'\Delta'''}{\Delta^2}-\frac{3}{2}\frac{\Delta''^2}{\Delta^2}+12\frac{\Delta'^2\Delta''}{\Delta^3}-\frac{13}{2}\frac{\Delta'^4}{\Delta^4}=0.)$$

The analytic fact expressed by this result has significant algebraic consequences, since it shows that if f is a non constant forms, then f and $\mathbf{D}f$, where $\mathbf{D}f=[f\,,\,f]_2$ in (1.1) are algebraically independent and therefore the full graded ring of automorphic forms for the given group is algebraic over $\mathbb{C}[f,\mathbf{D}f]$[20]. One notes that the foundation of Resnikoff's result was due to results of R.Rankin([18] and [19]);

Later, Resnikoff generalized the theorem of Hurwitz for every holomorphic automorphic forms on an irreducible bounded symmetric domain in the sense of É. Cartan. In particular, Resnikoff generalized the theorems of Hurwitz and Rankin for automorphic forms defined on finite dimensional products of half-planes[20] and, as a consequence, he was able to prove that any reasonably defined non constant automorphic form defined on the products of two half planes satisfies a differential equation of order 3[20].

Recently, non-existence of differential equation satisfied by Jacobi forms has been proved as an analogous theorem by Hurwitz as well as the existence of differential equation satisfied by Jacobi forms has been settled(see [6]). Namely, any non constant Jacobi form on the full Jacobi group satisfies a non-linear differential equation of order 3. The proof highly relies on the explicit form of Rankin-Cohen bilinear differential operators defined over the ring $J_{k,m}(,\,)$ of Jacobi forms which was studied in [2] and [5].

3. Further Research

There are correspondences among automorphic forms, Jacobi forms and Siegel forms[13]. One can show that Rankin-Cohen bilinear operators on each space are related in terms of theta-series expansion as well as Fourier-Jacobi expansions(see [2, 3]. One can also study analogous theories on the higher genus forms [7] as well as multi-linear holomorphic differential operators with arithmetic applications[4].

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