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# A SURVEY ON ELLIPTIC CURVES HAVING GOOD REDUCTION EVERYWHERE

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Abstract. An elliptic curve over a number field $K$ is said to have good reduction everywhere over $K$ if it has good reduction for every finite place $\nu$ of $K$. We give a brief survey on the theory of elliptic curves having good reduction everywhere over quadratic fields.

## Introduction

A well known theorem of Tate states that there is no elliptic curve over $\mathbf{Q}$ with good reduction everywhere. However there are many examples of elliptic curves over quadratic fields having good reduction everywhere. For example, the following curve given by Tate has good reduction everywhere over $\mathbf{Q}(\sqrt{29})$.

$$y^2 + xy + \epsilon^2 y = x^3,$$

where $\epsilon = \frac{5+\sqrt{29}}{2}$ and the discriminant $\Delta = -\epsilon^{10}$. As the discriminant shows, it is a global minimal Weierstrass equation. It is also well known that, if a number field $K$ is a principal ideal domain, then any elliptic curve over $K$ has a global minimal model. Let us briefly explain this. Let $E$ be an elliptic curve over $\mathcal{O}_K$, the ring of integers of a number field $K$,

$$E: \qquad y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

where each $a_i$ is in $\mathcal{O}_K$. Let $\Delta$ be the discriminant of $E$. For every finite place $\nu$ of $K$, we may think of a change of coordinates,

$$x = u_\nu^2 x_\nu + r_\nu, \quad y = u_\nu^3 y_\nu + s_\nu u_\nu^2 x_\nu + t_\nu, \quad u_\nu, s_\nu, t_\nu, r_\nu \in \mathcal{O}_K,$$

such that the corresponding curve is a minimal Weierstrass equation at $\nu$, that is, the discriminant $\Delta_\nu$ of new equation has the property that $ord_\nu(\Delta_\nu)$ is minimal. The minimal discriminant $\mathcal{D}$ of $E$ over $K$ is an integral ideal defined as $\mathcal{D} = \Pi_\nu \mathfrak{p}_\nu^{ord_\nu(\Delta_\nu)}$ where $\mathfrak{p}_\nu$ is the prime ideal associated to $\nu$ and the product runs through all finite places of $K$. ¿From the expression of a change of coordinates, one has $\Delta = u_\nu^{12}\Delta_\nu$ and it is clear that $(\Delta)/\mathcal{D}$ has the factor $\mathfrak{p}_\nu^{12 ord_\nu(u_\nu)}$ for every finite place $\nu$. Therefore there is an integral ideal $\mathcal{I}$ such that

$$(\Delta) = \mathcal{D}\mathcal{I}^{12}.$$

Above relation says that $E$ is a global minimal Weierstrass equation if and only if $\mathcal{I} = \mathcal{O}_K$. If a number field $K$ is a PID, then $\mathcal{I}$ is principal, which implies that after a change of coordinates, we have $(\Delta') = \mathcal{D}$ where $\Delta'$ is the new discriminant after

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the transformation. Thus $E$ has a global minimal model when $K$ is a PID. Suppose that an elliptic curve over $K$ has good reduction everywhere, then $\mathcal{D} = \mathcal{O}_K$ and we have $(\Delta) = \mathcal{I}^{12}$. Further assume that the class number of $K$ is prime to 6, then $\mathcal{I}$ should be a principal ideal and again after a change of coordinates, we have a global minimal model for $E$ over $K$. This is what Setzer showed in [10]. Namely,

**(Setzer).** *If a number field $K$ has the class number prime to 6 and if an elliptic curve $E$ over $K$ has good reduction everywhere, then $E$ has a global minimal Weierstrass equation.*

## Good reduction over imaginary quadratic fields

Let $k$ be an imaginary quadratic field and let $E$ be an elliptic curve over $k$. In his article [14], Stroeker proved that if $E$ has a global minimal model, then $E$ has bad reduction at $\nu$ for at least one finite place $\nu$ of $k$. The idea of Stroeker is quite natural to follow. He considered an elliptic curve over $k$ which is a global minimal Weierstrass equation and noticed the relation

$$c_4^3 - c_6^2 = 2^6 3^3 \Delta,$$

where $\Delta$ is the discriminant of $E$ and $c_4$ and $c_6$ are expressed in terms of $a_i$, the coefficients of $E$. (For the definition of $c_i$, refer any standard textbook on elliptic curves.) If $E$ has good reduction everywhere, then $(\Delta) = \mathcal{O}_k$ and we have a Diophantine equation,

$$c_4^3 - c_6^2 = \pm\epsilon 2^6 3^3, \quad \epsilon \in \mathcal{O}_k^*.$$

Stroeker showed that above Diophantine equation does not have an integral solution over $\mathcal{O}_k$, which gives a contradiction. One crucial advantage in the case of imaginary quadratic fields is the structure of the group of units of $k$. It is a finite group consisting of roots of unity and except the cases $k = \mathbf{Q}(\sqrt{-1}), \mathbf{Q}(\sqrt{-3})$, it is $\{\pm 1\}$. Therefore essentially one needs to solve $c_4^3 - c_6^2 = \pm 2^6 3^3$ over quadratic fields. But if one wants to apply the same technique to the case of real quadratic fields, one encounters much difficulties arising from the infinite cyclic group structure of units. Stating Stroeker's result again,

**(Stroeker).** *If an elliptic curve $E$ over an imaginary quadratic field $k$ has good reduction everywhere, then $E$ does not have a global minimal model.*

Combining above result with that of Setzer, we immediately get

**(Setzer,Stroeker).** *If an imaginary quadratic field $k$ has the class number prime to 6, then there is no elliptic curve over $k$ having good reduction everywhere.*

Above theorem explains the situation of *good reduction* over imaginary quadratic fields in a modest way. However it should be mentioned that some elliptic curves over certain imaginary quadratic field $k$ have good reduction everywhere. In fact there are infinitely many such $k$. For example the following is known. (See [14].)

**Theorem.** *let $n$ be an integer prime to 6 and suppose that $j$ satisfies $j^2 - 1728j \pm n^{12} = 0$. Then the elliptic curve*

$$y^2 + xy = x^3 - \frac{36}{j-1728}x - \frac{1}{j-1728}$$

*has good reduction everywhere over the quadratic field $\mathbf{Q}(j)$.*

For more examples, see [11].

## Good reduction over real quadratic fields

Much used notion here is so called *admissible curves*. An elliptic curve over $k$ is called admissible if it satisfies the following two conditions,

1. it has good reduction everywhere.
2. it has a $k$-rational point of order two.

In addition if it admits a global minimal model, then the curve is called $g$-admissible. Comalada [2], by solving some Diophantine equations explicitly, showed that there is an admissible elliptic curve over $k = \mathbf{Q}(\sqrt{m})$ $(1 < m < 100)$ if and only if $m = 6, 7, 14, 22, 38, 41, 65, 77, 86$. In addition he found all admissible curves up to isomorphism for these values of $m$ and made a table of them. The list of Comalada has instant application, in other words, it can be used to prove nonexistence of elliptic curve having good reduction everywhere over certain real quadratic field. As explained before, an admissible curve should satisfy two conditions, having good reduction everywhere and existence of $k$-rational point of order two. Over some real quadratic fields, the first condition automatically implies the second. Therefore if $k = \mathbf{Q}(\sqrt{m})$, $m$ less than 100, is such field, it should be in the list of quadratic fields which Comalada found. If it is not in the list, the first condition should not be satisfied, which implies that, over such field, there is no elliptic curve having good reduction everywhere. Kida and Kagawa used this idea to prove nonexistence of elliptic curve having good reduction everywhere over many real quadratic fields. Let $E$ be an elliptic curve having good reduction everywhere over real quadratic field $k$. If it does not have a $k$-rational point of order two, then we have a following inclusions,

$$k \subset k(\sqrt{\Delta}) \subset k(E[2]),$$

where the second inclusion is a cubic extension. Further if we assume that $E$ has a global minimal model (as is true when the class number of $k$ is prime to 6.) then $\Delta$ is a unit and we have limited choices of the field $k(\sqrt{\Delta})$. By prescribing certain conditions on the fundamental unit of $k$ and the class number of $k(\sqrt{\Delta})$, Kida proved that if $m = 2, 3, 5, 13, 17, 21, 47, 73, 94, 97$, then over $k = \mathbf{Q}(\sqrt{m})$ every elliptic curve having good reduction everywhere has a $k$-rational point of order two. However those $m$ are not in the list of Comalada. Therefore Kida have shown (see [6] [8].)

**(Kida).** *Over the quadratic fields $k = \mathbf{Q}(\sqrt{m})$, where $m = 2, 3, 5, 13, 17, 21, 47, 73, 94, 97$, there is no elliptic curve having good reduction everywhere.*

Kida also proved that when $m = 6, 7, 14, 41$, every elliptic curve over $k = \mathbf{Q}(\sqrt{m})$ having good reduction everywhere has a $k$-rational point of order two. But these $m$ do not appear in the list of Comalada, from which one can conclude that elliptic curves what Comalada found in his table are actually all the curves having good reduction everywhere in the cases $m = 6, 7, 14, 41$. When $k = \mathbf{Q}(\sqrt{37})$, by the works of Kagawa and Kida [5] [7], we also have an information of all elliptic curves having good reduction everywhere. (there are only two of them up to isomorphism over $k$.) The cases when $m = 6, 7, 14, 41$ and 37 provide partial information for the conjecture of Pinch relating so called Shimura's elliptic curves to elliptic curves having good reduction everywhere over real quadratic fields. Let us explain this. Let $N$ be a prime congruent to $1 \bmod 4$ and let $\chi_N$ be the Dirichlet character associated to the quadratic field $k = \mathbf{Q}(\sqrt{N})$. Define a map $\chi$ as follows.

$$\begin{array}{rccc} \chi : & \Gamma_0(N) & \longrightarrow & \{\pm 1\} \\ & \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & \chi_N(a) \end{array}$$

where $\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \mid c \equiv 0 \bmod N\}$. Let $X_0 = X_0(N)$ be the modular curve obtained from the compactification of $\mathcal{H}/\Gamma_0(N) \cup cusps$ and $J_0 = J_0(N)$ be the Jacobian of $X_0$. Letting $\Gamma_\chi = ker\,\chi$, we define the modular curve $X_\chi$ corresponding to the congruence subgroup $\Gamma_\chi$ and the Jacobian $J_\chi$ in the same way. By the work of Shimura, it is known that the abelian variety

$$J = coker\,(J_0 \to J_\chi)$$

is of even dimension and defined over $\mathbf{Q}$. Furthermore it splits over $k = \mathbf{Q}(\sqrt{N})$ as $J = B \times B^\sigma$ up to isogeny where $\sigma$ is the galois conjugation of $k$. Shimura also showed that $B$ has good reduction everywhere and $B$ is isogenous to $B^\sigma$ over $k$. When $J$ is of dimension two, $B$ is an elliptic curve (which is usually called Shimura's elliptic curve) having good reduction everywhere over $\mathbf{Q}(\sqrt{N})$. Pinch [9] conjectured that if an elliptic curve over a real quadratic field $k = \mathbf{Q}(\sqrt{N})$ has good reduction everywhere and is isogenous to its galois conjugate (i.e. $\mathbf{Q}$-curve), then it shoud be isogenous to Shimura's elliptic curve. When $N = 29, 41$ and $37$, this conjecture is known to be true. For example when $N = 37$, the abelian variety $J$ is of dimension two and $J = B_{37} \times B_{37}^\sigma$ where

$$B_{37}: \quad y^2 - \epsilon y = x^3 + \frac{3\epsilon+1}{2}x^2 + \frac{11\epsilon+1}{2}x, \quad \Delta = \epsilon^6, j = 2^{12},$$

and $\epsilon = 6 + \sqrt{37}$ is the fundamental unit. Since one can easily show that $P_0 = (0,0)$ is a point of order five, there is a 5-isogeny between $B_{37}$ and the following curve,

$$B_{37}/\langle P_0 \rangle : \quad y^2 - \epsilon y = x^3 + \frac{3\epsilon+1}{2}x^2 - \frac{1669\epsilon+139}{2}x - 7(5449\epsilon + 451),$$

where $\Delta = \epsilon^6$ and $j = 3376^3$. Each of above curves is isomorphic to its galois conjugate and they are all the curves over $\mathbf{Q}(\sqrt{37})$ having good reduction everywhere.

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