Trends in Mathematics
Information Center for Mathematical Sciences
Volume 1, Number 1, September 1998, Pages 37–40

ON REGULAR BAER RINGS

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ABSTRACT. We show that a countable π-regular Baer ring is semiprimary. Actually it is shown that a π-regular Baer ring with only countably many idempotents is semiprimary. This rersult allows us to generalize a theorem of Rangaswamy on regular Baer rings. Examples are provided to illustrate and delimit our results.

R will denote an associative ring with unity, $J(R)$ its Jacobson radical and $\mathbf{I}$(R) its set of idempotent elements. We use $|X|$ and $\mathbf{c}$ to denote the cardinality of a set X and the cardinality of the continuum, respectively.

¿From [4], a ring R is called π-regular if for each element $a \in R$ there exist a positive integer n (depending on a) and an element $x \in R$ such that $a^n = a^n x a^n$. A π-regular ring R for which the n in the above can be taken to be 1 is called *regular*. Recall from [5] that a ring R is *Baer* if the right annihilator of every nonempty subset of R is generated as a right ideal by an idempotent. The study of Baer rings has its roots in functional analysis [5]. A ring R is called *right* (resp. *left*) *PP* if the right (resp. left) annihilator of every element of R is generated as a right (resp. left) ideal by an idempotent. Note that every Baer ring is a right and left PP ring.

In 1950 [4], Kaplansky proved that every orthogonally finite (i.e., no infinite set of pairwise orthogonal idempotents) π-regular ring is semilocal. In 1967 [8], Small proved that every orthogonally finite right PP ring is Baer. As a corollary he obtained that every right perfect right PP ring is semiprimary and left PP. In 1974 [7], Rangaswamy established that a regular Baer ring of cardinality less than $\mathbf{c}$ is semisimple Artinian.

The foregoing results motivated us to ask the following natural question: *Is a countable π-regular Baer ring a semiprimary ring?* In this paper we give a positive answer to this question. More generally we will show that if R is a π-regular Baer ring with $|\mathbf{I}(R)| < \mathbf{c}$, then $R = A \oplus B$ where A is a finite direct sum of division rings A_i, where $|A_i| \geq \mathbf{c}$ and B is a semiprimary ring with $|B| < \mathbf{c}$. Rangaswamy's theorem then becomes an immediate corollary of this result.

Recall from [3, p.210] that a ring R is called an *I-ring* if every nonnil right ideal of R contains a nonzero idempotent. It can be checked easily that π-regular rings are I-rings.

Theorem 1. *Let R be an orthogonally finite right PP ring. Then the following conditions are equivalent:*

(i) *R is an I-ring;*

1991 *Mathematics Subject Classification.* 16E50,16D30.

Key words and phrases. von Neumann regular ring, Baer ring, π-regular ring and orthogonally finite ring.

(ii) *R is a semiprimary ring;*
(iii) *R is a π-regular ring.*

Proof. Since semiprimary rings are π-regular and π-regular rings are I-rings, we only need to prove (i) $\Rightarrow$(ii). Let R be an orthogonally finite I-ring which is right PP. Then by [4, Theorem 2.1] R is semilocal. So it is enough to show that $J(R)$ is nilpotent, where $J(R)$ is the Jacobson radical of R. Write $R = \sum_{i=1}^{n} e_iR$, where $\{e_1, \cdots, e_n\}$ is a complete set of primitive idempotents. Now $J(R) = e_1J(R) + \cdots + e_nJ(R)$. We claim that each $e_iJ(R)$ is nilpotent. Suppose that there exists some $e_iJ(R)$ which is not nilpotent. Then $(e_iJ(R))^2 \neq 0$ for some i, so $e_ixe_i \neq 0$ for some $x \in J(R)$. Consider the map $f_{e_ix} : e_iR \to e_iR$ defined by $f_{e_ix}(e_ir) = e_ixe_ir$. So $\mathrm{Im}(f_{e_ix})$ is projective. Thus $\mathrm{Ker}(f_{e_ix})$=0 since e_iR is indecomposable and $\mathrm{Im}(f_{e_ix}) \neq 0$. Now $e_ix \in J(R)$. Since R is an I-ring, $J(R)$ is nil! and hence there is an integer $m > 1$ such that $(e_ix)^m = 0$ and $(e_ix)^{m-1} \neq 0$. But $(e_ix)^{m-1} \in$ Ker $(f_{e_ix}) = 0$, which is a contradiction. Thus each $e_iJ(R)$ is nilpotent. Consequently $J(R)$ is nilpotent.

Immediately we have the following corollary which is due to Small [8].

Corollary 2. *If R is a right perfect right PP ring, then R is a semiprimary ring.*

Example 3. The condition "orthogonally finite" in Theorem 1 is not superfluous. There exists a Baer ring (hence right PP) which is an I-ring, but not π-regular. Now let $R = \{(a_n)_{n=1}^{\infty} \in \prod \mathbb{Q} \mid a_n \in \mathbb{Z} \text{ eventually}\}$, where $\prod \mathbb{Q}$ is the countably infinite direct product of the rationals $\mathbb{Q}$. Then $\prod \mathbb{Q}$ is the maximal ring of quotients $Q(R)$ of R . Since $Q(R)$ is regular self-injective, it is a Baer ring. Also note that the set of all idempotents of $Q(R)$ is that of R. Now for a nonempty subset X of R, it follows that $r_R(X) = r_{Q(R)}(X) \cap R = eQ(R) \cap R$ for some idempotent $e \in R$. Therefore $r_R(X) = eR$, and hence R is a Baer ring. Next, to show that R is an I-ring, let K be a nonzero ideal of R. Then there is a nonzero element, say $x \in K$ with nonzero k-th coordinate, say x_k for some k. Let $y \in Q(R)$ with the k-th coordinate x_k^{-1} and 0 for the other coordinates. Then $y \in R$ and $xy \in K$ is a nonzero idempotent in R. So R is an I-ring. Finally, let $\alpha = (2, 2, \dots) \in R$. If R is π-regular, then there are a positive integer n and an element $\beta \in R$ such that $\alpha^n = \alpha^n\beta\alpha^n$. So there is an integer m such that $2^n = 2^n m 2^n$, which is a contradiction. Thus the ring R cannot be π-regular.

Recall the following result due to Rangaswamy [7,Theorem 1] : Any countable regular Baer ring is semisimple Artinian. In order to give a nontrivial generalization of this result we have the following preparatory lemma.

Lemma 4. *If R is a Baer ring with $|\mathbf{I}(R)| < \mathbf{c}$, then R is orthogonally finite.*

Proof. See [6, Theorem 2].

Theorem 5. *Let R be a π-regular Baer ring with $|\mathbf{I}(R)| < \mathbf{c}$. Then R is semiprimary. In particular, if R is semiprime, then R is semisimple Artinian.*

Proof. This result is a consequence of Lemma 4 and Theorem 1.

Corollary 6. *Let R be a π-regular Baer ring with $|\mathbf{I}(R)| < \mathbf{c}$. Then*
(i) *$R = A \oplus B$ (ring direct sum) and B is countable;*
(ii) *$A = \oplus_{i=1}^{m} A_i$, where each A_i is a division ring with $|A_i| \geq \mathbf{c}$;*

(iii) $B \cong \begin{pmatrix} B_1 & B_{12} & \cdots & B_{1k} \\ 0 & B_2 & \cdots & B_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_k \end{pmatrix}$, *where each* B_i *is a simple Artinian ring and each* B_{ij} *is a left* B_i*-right* B_j*-bimodule.*

Furthermore, if R *is* π*-regular Baer ring with* $|\mathbf{I}(R)| < \mathbf{c}$ *such that* R *is an algebra over an uncountable field, then* R *is a finite direct sum of division rings.*

Proof. Since R is semiprimary Baer by Theorem 5, it follows that there is a positive integer n by [1,Theorem 4.4] such that

$$R \cong \begin{pmatrix} R_1 & R_{12} & \cdots & R_{1n} \\ 0 & R_2 & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_n \end{pmatrix}$$

with each R_i is simple Artinian and each R_{ij} is a left R_i-right R_j-bimodule. For $1 \le i \le n-1$, if R_i is uncountable, then R_i should be a division ring. In this case, since R_{ij} for $i < j \le n-1$ is a left vector space over a division ring R_i, it follows that $R_{ij} = 0$ whenever R_i is uncountable. Next, suppose that R_n is uncountable. Then it also should be a division ring. In this case, since R_{in} for $1 \le i \le n-1$ are right vector spaces over R_n. Thus each $R_{in} = 0$ whenever R_n is a division ring. Thus we can get desired result.

The following result of Rangaswamy [7] becomes an immediate consequence of Theorem 5 or Corollary 6.

Corollary 7. *Let* R *be a regular Baer ring with* $|R| < \mathbf{c}$. *Then* R *is semisimple Artinian.*

The following examples are provided to delimit our results. Recall that a ring R is called *right* (resp. *left*) *weakly regular* (or *fully idempotent*) if $a \in aRaR$ (resp. $a \in RaRa$) for every $a \in R$. Right and left weakly regular rings are called *weakly regular*. Recall [2] that R is a *quasi-Baer* if the right annihilator of every right ideal of R is generated (as a right ideal) by an idempotent.

By Theorem 5, regular Baer rings with only countably many idempotents are semisimple Artinian. So one may raise the following question: *Is a weakly regular Baer ring or a regular quasi-Baer ring with only countably many idempotents a semisimple Artinian ring?* But the following two examples eliminate these possibilities.

Example 8. There exists a weakly regular Baer ring with only countably many idempotents, but it is not semisimple Artinian. In fact let R be the first Weyl algebra over a field of characteristic zero. Then R is a weakly regular Baer ring with only countably many idempotents, but R is not semisimple Artinian.

Example 9. There exists a regular quasi-Baer ring with only countably many idempotents, but it is not semiprimary. Thus Theorem 5 cannot be extended to the case of quasi-Baer rings. For a finite field F, let

$$R = \{ \begin{pmatrix} A & & & 0 \\ & a & & \\ & & a & \\ 0 & & & \ddots \end{pmatrix} \mid A \in \mathrm{Mat}_n(F), a \in F, n = 1, 2, \dots \}.$$

Then R is prime regular, so R is quasi-Baer. Also note that R is countable. However R is not orthogonally finite and hence R cannot be semiprimary. In this case, note that R cannot be a Baer ring because of Theorem 5.

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