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A SURVEY ON TOURNAMENT MATRICES (FROM THE COMBINATORIAL MATRIX THEORETIC VIEWPOINT)

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ABSTRACT. A tournament is the record of a round-robin competition between players. It is combinatorially represented by a digraph whose adjacency matrix is called a tournament matrix. Tournament matrices enjoy many interesting combinatorial properties. They can be studied combinatorial matrix theoretically as well as graph theoretically. From the combinatorial matrix theoretic viewpoint, some interesting recent results on them are summarized, and some problems are mentioned.

1. INTRODUCTION

A *tournament* T on the vertices $\{1, 2, \ldots, n\}$ is a loop-free directed graph with the property that for any pair of distinct vertices i and j, T contains either the arc $i \to j$ or the arc $j \to i$. When T contains the arc $i \to j$, we say that i defeats j or i dominates j.

A *tournament matrix* M is the $(0, 1)$ adjacency matrix of a tournament T and it satisfies the equation $M + M^t = J - I$, where J is the all ones matrix and I is the identity matrix. An $n \times n$ tournament matrix can be thought of as a record of the results of a round robin competition between n players.

A square matrix A is *reducible* provided that there exists a permutation matrix P such that PAP^t is of the form $\begin{bmatrix} B & 0 \\ C & D \end{bmatrix}$, where B and D are nonvacuous square matrices and 0 is the zero matrix. The matrix A is *irreducible* provided that it is not reducible. Equivalently, A is irreducible if and only if the digraph associated with A is strongly connected.

In the following sections, we briefly look at spectral properties, ranking problems, arc reversal problems, and some generalizations of tournament matrices, and state some related problems.

2. SPECTRAL PROPERTIES OF TOURNAMENT MATRICES

Brauer and Gentry [1] have noticed the fact that for an eigenvalue λ of a tournament matrix of order n, $-1/2 \le \mathrm{Re}\lambda \le (n-1)/2$ and the spectral properties of tournament matrices have been of much interest and studied by many thereafter.

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Let $\mathrm{var}(x) = \sum_{i<j} |x_i - x_j|^2$ denote the variation of a vector $x = (x_1, \ldots, x_n)^t$. Then using the identity $\mathrm{var}(x) = n|x|^2 - |\sum_{i=1}^n x_i|^2$ and pre- and post-multiplying $M + M^t = J - I$ by x^* and x we easily obtain the following theorem:

Theorem 2.1. [16] *Let M be a tournament matrix and let λ and x be, respectively, an eigenvalue and the eigenvector corresponding to λ of M. Then*

(a)

$$\frac{2\mathrm{Re}\lambda}{n-1} + \frac{\mathrm{var}(x)}{(n-1)|x|^2} = 1,$$

(b) $\mathrm{Re}\lambda \leq (n-1)/2$ *with equality holding if and only if* $\mathrm{var}(x) = 0$,

(c) $\mathrm{Re}\lambda \geq -1/2$ *with equality holding if and only if* $\sum_{i=1}^n x_i = 0$.

For an irreducible tournament matrix M, the Perron-Frobenius theorem [5] guarantees the existence of an algebraic simple positive eigenvalue ρ which is equal to the maximum of the moduli of the eigenvalues; ρ is called the Perron value for M. Let v be a right Perron vector for M. Then $\rho = (n-1)/2$ if and only if $M\mathbf{1} = ((n-1)/2)\mathbf{1}$ where $\mathbf{1}$ is the all ones column vector, and such a matrix is called a regular tournament matrix, i.e., M has the same row sums. We call $s = M\mathbf{1} = (s_1, \ldots, s_n)^t$ the score vector of M. Each s_i denotes the number of games player i has won, i.e., s_i is the outdegree of vertex i in the tournament T associated with M. So M is regular if and only if $s = ((n-1)/2)\mathbf{1}$.

The following theorem has been observed by many.

Theorem 2.2. *Let M be a tournament matrix of order n. If λ is an eigenvalue of M, then its geometric multiplicity is 1 whenever* $\mathrm{Re}\lambda \neq -1/2$. *If* $\mathrm{Re}\lambda = -1/2$, *then the geometric multiplicity and the algebraic multiplicity of λ are the same.*

It is easy to verify that an $n \times n$ tournament matrix M is regular if and only if M is normal. So there exists the basis consisting of n linearly independent eigenvectors for M so that M can be diagonalizable. In fact, let $\rho = (n-1)/2, \lambda_1, \ldots, \lambda_{n-1}$ be the eigenvalues of a regular tournament matrix M. Then since $\mathrm{tr}M = \rho + \sum_{i=1}^{n-1} \mathrm{Re}\lambda_i = 0$ and each $\mathrm{Re}\lambda_i \geq -1/2$, we have $\mathrm{Re}\lambda_i = -1/2$ for all $1 \leq i \leq n-1$. In particular, the rank of a regular tournament matrix is n.

Since an irreducible tournament matrix M of order $n \geq 3$ satisfies $\mathrm{tr}M^2 = 0$, M must have at least three distinct eigenvalues including its Perron value ρ.

Theorem 2.3. [4] *An irreducible tournament matrix M of order $n \geq 3$ has exactly three eigenvalues if and only if M is a Hadamard tournament matrix of order n, i.e., M satisfies*

$$MM^t = \frac{n+1}{4}I + \frac{n-3}{4}J \quad (n \equiv 3 \ (mod\ 4)).$$

A Hadamard tournament matrix has eigenvalues $(n-1)/2$ and $-1/2 \pm i\sqrt{n}/2$ and hence we see that the maximum geometric multiplicity of an eigenvalue with its real part equal to $-1/2$ for an irreducible tournament matrix is $(n-1)/2$.

Theorem 2.4. [14] *An irreducible singular tournament matrix M with 0 as a simple eigenvalue has at least four distinct eigenvalues. Furthermore, M has exactly four distinct eigenvalues if and only if $J - 2M$ is a skew Hadamard matrix, i.e.,*

$B = J - 2M$ is a $(1, -1)$ matrix which satisfies $B^tB = nI$ and $B - I$ is a skew-symmetric matrix.

In fact, the rank of a tournament matrix is at least $n-1$ and Maybee and Pullman [15] conjectured that for sufficiently large n, almost all tournament matrices of order n are nonsingular. Shader [18] and Kirkland [8] proved that if the score vector satisfies $s^ts < n^2(n-1)/4$, then M is nonsingular. Shader also proved that given $\varepsilon > 0$, for sufficiently large n, the fraction of tournament matrices of order n whose determinant has the sign equal to $(-1)^{n-1}$ is at least $1/2 - \varepsilon$.

Let π_n denote the minimum Perron value over the set of all irreducible tournament matrices of order n. Brauer and Gentry [2] showed that $\sqrt{2} < \pi_n$ for all $n \geq 4$, and de Caen et al. [4] showed that $2 < \pi_n < 2.5$ for all $n \geq 8$. Brualdi and Li [3] conjectured that the irreducible tournament matrix of order n which minimizes Perron value is of the following form

$$\begin{bmatrix} 0 & 1 & & & & \\ 0 & 0 & 1 & & 0 & \\ & & \ddots & & & \\ & & & \ddots & & \\ & 1 & & 0 & 0 & 1 \\ & & & & 0 & 0 \end{bmatrix}$$

and the tournament matrix of even order with largest Perron value can be described as

$$\begin{bmatrix} T & T^t \\ T^t + I & T \end{bmatrix},$$

where T is the matrix of order $n/2$ with zeros on and below the main diagonal and ones above the main diagonal.

Kirkland [11] proved the former by investigating score vectors and using arc reversal method, and [12] proved the weaker version of the latter conjecture: for all sufficiently large n, a tournament matrix of order n with the maximum Perron value is almost regular, i.e., tournament matrix of even order n with the half of row sums equal to $(n-2)/2$ and the other half equal to $n/2$.

3. Ranking Problems

As mentioned above, the score vector $s = M\mathbf{1} = (s_1, \ldots, s_n)^t$ of a tournament matrix M provides a natural way to rank the players $\{1, 2, \ldots, n\}$ since s_i reads the number of games that player i has won. When M is an irreducible tournament matrix, Kendall-Wei ranking scheme interprets the entries in the right Perron vector w of M as the strengths of the players, so that player i is ranked higher than player j if and only if $w_i > w_j$. Theorem 2.1(a) implies that $\mathrm{var}(w)/|w|^2$ or the Perron value ρ itself is a measure of the overall closeness of the ranking of the players under the Kendall-Wei scheme. So if M is a regular tournament matrix, $\rho = (n-1)/2$ and all players are evenly ranked.

Problem : *Under what circumstances does the Kendall-Wei ranking agree with the ranking of the players according to the row sums of M?*

When M is a regular tournament matrix, it is easy to see that two rankings agree. Kirkland [12] showed that for almost regular tournament matrices, two rankings also agree.

4. Arc Reversal Problems

Given a tournament matrix M, its *reversal index*, $i_r(M)$, is the minimum k such that the reversal of the orientations of k arcs in the digraph associated with M results in a reducible matrix.

Kirkland [9] established the inequality $i_r(M) \leq \lfloor (n-1)/2 \rfloor$ with equality holding if and only if M is regular or almost regular according to n is odd or even, and constructed the tournament matrices of order n whose reversal index is k for each k with $1 \leq k \leq \lfloor (n-1)/2 \rfloor$.

Problem : *Given M, how can we identify the sets S of arcs such that $M(S)$ is reducible, where $M(S)$ denotes the tournament matrix obtained from M by reversing the orientations of the arcs in S.*

5. Multipartite Tournament Matrices with Constant Team Size

Let $p_1, \ldots, p_d$ be positive integers. A digraph obtained by orienting each edge of the complete d-partite graph $K_{p_1,\ldots,p_d}$ is a *d-partite tournament.* A digraph which arises from orienting the edges of $K_{l,\ldots,l}$ is a *d-partite tournament with team size* l [13]. The name derives from considering a competition between d teams of l players each, in which each pair of players from different teams compete in exactly one game.

We may assume that kth team consists of the players in the set $\{(k-1)l + 1, \ldots, kl\}$, $(1 \leq k \leq d)$. Then the adjacency matrix M of the d-partite tournament with team size l satisfies $M + M^t = (J_d - I_d) \otimes J_l = J_n - (I_d \otimes J_l)$, where $\otimes$ denotes the Kronecker product of matrices.

Let $\mathcal{T}_{d,l}$ be the set of all d-partite tournament matrices with team size l. We have analogous results to the case of tournament matrices:

Theorem 5.1. *Suppose $M \in \mathcal{T}_{d,l}$ and that λ is an eigenvalue of M with corresponding eigenvector v. Then*

(a) $-l/2 \leq \mathrm{Re}\lambda \leq (n-l)/2$,
(b) $\mathrm{Re}\lambda = (n-l)/2$ *if and only if M is regular, i.e., $M\mathbf{1} = ((n-l)/2)\mathbf{1}$,*
(c) $\mathrm{Re}\lambda = -l/2$ *if and only if $v^* J_n v = 0$ and for $1 \leq i \leq d$, each $v^{(i)} = (v_{(i-1)l+1}, \ldots, v_{il})^t$ is a multiple of $\mathbf{1}_l$.*

Theorem 5.2. *Suppose $l \geq 2$ and that $M \in \mathcal{T}_{d,l}$ is irreducible. Then M has at least four distinct eigenvalues.*

This theorem implies that the algebraic multiplicity of a nonnegative eigenvalue λ of an irreducible matrix $M \in \mathcal{T}_{d,l}$ is at most $(n-2)/2$.

Theorem 5.3. *If $M \in \mathcal{T}_{d,l}$ and λ is an eigenvalue of M with $\mathrm{Re}\lambda = -l/2$, then the geometric multiplicity and the algebraic multiplicity of λ are equal.*

Let $M \in \mathcal{T}_{d,l}$ be regular and suppose M has k purely imaginary eigenvalues. Let the remaining eigenvalues of M be $(n-l)/2$, $\lambda_1, \ldots, \lambda_{n-1-k}$. Since $\mathrm{tr}M =$

$(n-l)/2 + \sum_{i=1}^{n-1-k} \mathrm{Re}\lambda_i = 0$ and $\mathrm{Re}\lambda_i \geq -l/2$ for all i, we have $k \leq n-d$. This means that the rank of a regular matrix in $\mathcal{T}_{d,l}$ is at least d. We can say more:

Theorem 5.4. *Let $M \in \mathcal{T}_{d,l}$ be regular. If M has rank d, then d is odd. If d is even, then M has rank at least $d+1$.*

With the theorems in mind one may ask the following questions:

Problem : *For which values of d and l does there exist a matrix in $\mathcal{T}_{d,l}$ with exactly four distinct eigenvalues?*

Problem : *What is the maximum possible geometric multiplicity of an eigenvalue with real part equal to $-1/2$ for a matrix in $\mathcal{T}_{d,l}$?*

Problem : *If d and l are both even, what is the minimum possible rank of a regular matrix in $\mathcal{T}_{d,l}$?*

6. Extensions

Some extensions of tournament matrices have been studied. Moon and Pullman [16] considered *generalized tournament matrices*, those square nonnegative real matrices P satisfying $P + P^t = J - I$. They interpret the entry p_{ij} as a priori probability that player i defeats player j. Maybee and Pullman [15] looked at several extensions: a *pseudo-tournament matrix* which is a complex matrix M such that $M + M^* + I$ has rank one; a *hyper-tournament matrix* which is a real pseudo-tournament matrix with main diagonal entries equal to zero.

They characterized the singular hyper-tournament matrices and the vectors that can be in the kernels of such matrices. For example, they proved the following:

Theorem 6.1. *Singular, irreducible tournament matrices of order n exist if and only if $n \notin \{2,3,4,5\}$.*

Theorem 6.2. *For every $h \in R^n$ with $h_i^2 = 1$ for all i, there exist a singular irreducible h-hypertournament matrix M, i.e., M satisfies $M + M^t = hh^t - I$, if and only if $n \neq 2$.*

Kirkland [8] proved that if M is a generalized tournament matrix with score vector s satisfying $s^t s < n^2(n-1)/4$ then the spectral radius of M is larger than $(n-2)/2$.

Problem : *Is the converse true?*

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