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SOME RECENT TOPICS IN FANO MANIFOLDS

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ABSTRACT. This is a survey of geometric aspects of the study of Fano manifolds. In particular, some recent results from a joint program with N. Mok are discussed. The main tool is the deformation theory of rational curves originating from the work of Mori. The essential feature of our program is a systematic use of the variety of minimal rational tangents. A brief discussion of other aspects of Fano manifold theory will be given at the end.

A smooth algebraic variety X (over an algebraically closed field of characteristic zero) is a Fano manifold if its anticanonical bundle K_X^{-1} is ample. Many well-known classical algebraic varieties are Fano manifolds. Here are some examples.

1. Smooth hypersurfaces of degree $< n+2$ in the projective space $\mathbf{P}_{n+1}$. Although these are the 'simplest' algebraic varieties we learnt in the first class of algebraic geometry, there are many open questions on them.
2. Rational homogeneous spaces, namely, the coset spaces G/P for a connected semisimple algebraic group G and a parabolic subgroup P. These include the projective space $\mathbf{P}_n$, the Grassmannian, and the smooth quadric, i.e., the hypersurface in the projective space defined by a nondegenerate quadratic equation. These play a major role in the theory of algebraic groups. Their geometry is more well-understood than other varieties, which also means that there are more questions we can ask about them than other varieties.
3. Moduli spaces of vector bundles of rank r with a fixed determinant of degree d over an algebraic curve of genus ≥ 2 are Fano manifolds if $(d, r) = 1$. Although not as old as above two, these have been studied from the 1960's. The study of these Fano manifolds mixes the theory of curves with that of higher dimensional varieties. They are also important in topology and physics.

Why do people study Fano manifolds? Of course, the above examples show that these appear in many areas of mathematics, and they had been studied for quite a while even before we define the notion of "K_X^{-1} is ample". It was G. Fano who realized that many properties of these classically studied varieties come from the ampleness of the anticanonical bundle and thus it is worth studying a class of varieties with ample anticanonical bundles, thus the name "Fano manifolds". From the view-point of Mori theory, Fano varieties are the building blocks of general smooth varieties from its minimal models. This is one of the main reason that many algebraic geometers of 1990's are interested in them.

How many Fano manifolds are there? It is known that for a given dimension, there are only finitely many families of Fano manifolds ([KMM]). This means that

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in principle, we can classify all Fano manifolds of a given dimension. In dimension 2 and 3, the classification was done. But in dimension > 3, such classification seems to be hopelessly complicated. So for higher dimensions, people try to classify certain special classes of Fano manifolds. Some of these classes are defined by some numerical invariants of Fano manifolds. One basic invariant is the second Betti number b_2, which is equal to the rank of the Picard group for Fano manifolds. A Fano manifold X with $b_2 > 1$ allows a Mori contraction, meaning a regular map to another algebraic variety which is simpler than X in a suitable sense. So it is more essential to study Fano manifolds with $b_2 = 1$. In this case, the anticanonical bundle K_X^{-1} is an integer multiple of the ample generator of $Pic(X) \cong \mathbf{Z}$. This integer is called the index of X. It is known that the index is always less than or equal to $n+1, n = \dim(X)$. If the index is $n+1$, X must be the projective space, and if the index is n, X must be the hyperquadric. For lower indices, there are many possibilities. People have been trying to classify Fano manifolds of large indices.

Another important invariant is the length. In the celebrated paper [Mr], Mori showed that there exists a rational curve (i.e. an image of $\mathbf{P}_1$) through each point of a Fano manifold. The minimal degree of rational curves through generic points of a Fano manifold X with respect to the line bundle K_X^{-1} is called the length of X. Mori showed that the length is between 2 and $n+1$. It has been conjectured that Fano manifolds of length $n+1$ are projective spaces.

For a detailed presentation of all the foundational materials, see [Ko1]. Also surveys by Miyaoka [Mi1], [Mi2] are recommended.

My recent research concerns certain geometric problems on Fano manifolds. This is a joint project with N. Mok. Our starting point is the rational curves of minimal degree through a generic point of the Fano manifold studied by Mori in [Mr]. The main feature of our program is so-called the variety of minimal rational tangents. Let $x \in X$ be a generic point of the Fano manifold. We introduce a special subvariety $\mathcal{C}_x$ in the projectivized tangent space $\mathbf{P}T_x(X)$ as the closure of the union of tangent vectors to rational curves of minimal degree through x. This $\mathcal{C}_x$ is called the variety of minimal rational tangents. The dimension of $\mathcal{C}_x$ is exactly the length of X minus 2. Our philosophy is that the structure of X is determined by the projective geometry of $\mathcal{C}_x$ and how it varies as x changes. The following result will give an idea of what we have in mind.

Theorem ([HM1]) *Let X be a Fano manifold. Suppose there exist two vector bundles V, W of rank > 1 such that $T(X) \cong V \otimes W$. Then X is the Grassmannian.*

The key point of the proof is to note that under the assumption, rational curves of minimal degree on X must be tangent to the direction of pure tensors in $T_x(X) \cong V_x \otimes W_x$. A similar charcterization of Hermitain symmetric spaces, which is a special class of rational homogeneous spaces, is given in [HM1]. Hermitian symmetric spaces are the first objects that we applied this view-point. As a matter of fact, the variety of minimal rational tangents is a generalization of the characteristic variety for Hermitian symmetric spaces first used by Mok in [Mk]. The need for this general notion comes from our study of the deformation of Hermitian symmetric spaces:

Theorem ([HM2]) *Let $\mathcal{X}$ and $\mathcal{Y}$ be two connected algebraic varieties and $f : \mathcal{X} \to \mathcal{Y}$ be a smooth proper morphism. If for a point $y_o \in \mathcal{Y}$, the fiber $f^{-1}(y_o)$*

is isomorphic to the Grassmannian, then for any other point $y \in \mathcal{Y}$*,* $f^{-1}(y)$ *is isomorphic to the Grassmannian.*

This is proved by studying how the variety of minimal rational tangents changes as the Fano manifold is deformed. Here again, the statement holds with the word Grassmannian replaced by any irreducible Hermitian symmetric space. In fact, it holds for some other class of rational homogeneous spaces, too ([Hw1]). However it has not been proved for all rational homogeneous spaces of $b_2 = 1$.

One can define a subvariety of $\mathbf{P}T_x(X)$ just as the variety of minimal rational tangents using curves of higher genus instead of rational curves. The precise definition is more involved than the case of rational curves because deforming higher genus curves may change the (geometric) genus. A subvariety of $\mathbf{P}T_x(X)$ defined in this way is called the variety of distinguished tangents. They are useful in the study of finite morphisms between Fano manifolds. The basic result is the following.

Theorem ([HM5]) *Let* $f : X \to Y$ *be a finite morphism between Fano manifolds of the same dimension. For a sufficiently generic point* $x \in X$*,* $df^{-1}(\mathcal{C}_{f(x)})$ *is a variety of distinguished tangents.*

There are many applications of this theorem. The next theorem was a conjecture of Lazarsfeld.

Theorem ([HM5]) *Let* G/P *be a rational homogeneous space of* $b_2 = 1$*. Let* $f : G/P \to X$ *be a surjective morphism where* X *is a smooth variety. Then either* X *is a projective space or* f *is an isomorphism.*

In this case, X must be a Fano manifold. This theorem was proved by studying the inverse image of varieties of minimal rational tangents of X. I expect that this technique will be useful in the study of finite morphisms between Fano manifolds. There are many open questions in this direction. For example, is there a nontrivial morphism between smooth hypersurfaces of the same degree in $\mathbf{P}_{n+1}$?

There are many other applications of varieties of minimal rational tangents. See the survey articles [HM3] and [HM4] for further examples.

Finally, I would like to mention some of other active areas of research on Fano manifolds.

(1) Birational problems.

This is concerning the structure of the field of rational functions on Fano manifolds. Some of the most notorious problems of algebraic geometry are here. For example, is the fraction field of the domain $\mathbf{C}[x_0, \ldots, x_n]/ < f >$, where $f(x_0, \ldots, x_n)$ is a general polynomial of degree 4, isomorphic (as fields) to the rational function field $\mathbf{C}(x_1, \ldots, x_n)$? For an introductory survey, see [Ko2].

(2) Arithmetic of Fano manifolds.

The most well-known problem here is Manin's question about the growth of number of rational points of bounded height on a Fano manifold defined over a number field. See [FMT] and [Ma].

(3) Differential geometry.

A famous question of Calabi asks for an algebraic geometric condition on a Fano manifold which is equivalent to the differential geometric condition that the Fano manifold has an Einstein metric. Many sufficient conditions and necessary conditions are known. For a most recent result, see [Ti]. It is interesting to note that the multiplier ideal sheaf and Nadel's vanishing theorem, which since become popular tools in algebraic geometry, first appeared in connection with this problem ([Na]). This problem also gives rise to many algebraic geometric questions. One of them is the stability of the tangent bundles of Fano manifolds with $b_2 = 1$, which is a necessary condition for the existence of an Einstein metric. See [Hw2] and [PW].

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