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LORENTZIAN LATTICES AND $K3$ SURFACES

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Abstract. The main tools for the study of $K3$ surfaces are Torelli theorem and the theory of lattices. In this note, we briefly explain these important stuff and then proceed to introduce recent progress on automorphisms of $K3$ surfaces.

0. Introduction

Let X be a compact complex manifold (simply called a *surface*). Then $H^2(X, \mathbf{Z})$ torsion has a canonical structure of a lattice with the cup product. For cohomology classes dual to divisors, this cup product is nothing but the intersection number of divisors. It is not to much to say that studying $K3$ surfaces amounts to studying lattices. In 1910 Severi [19] showed the existence of a $K3$ surface with an infinite group of automorphisms by using the theory of intersections. However the author believe that it were Piatetskii-Shapiro and Shafarevich [18] who first established the direction of the study. They proved a Torelli-type theorem for algebraic $K3$ surfaces, where they essentially used the theory of lattices and discrete reflection groups. On the other hand, Nikulin [13] made a deep study of the theory of integral quadratic forms and made full use of it to study $K3$ surfaces [14]-[17]. After that, following their method, many results have been obtained. In this note we shall deal only with recent progress on automorphisms of $K3$ surfaces, including an application of the result of Conway [4,5] and Borcherds[1,2] on geometry of the extended Leech lattice.

1. $K3$ surfaces

$K3$ surfaces, classically known (in the last centry) as quartic surfaces in $\mathbf{P}^3$ or Kummer surfaces, were named so by A. Weil "because of Kummer, Kähler, Kodaira and of the beautiful mountain $K2$ in Cashmere". It is an interesting and fundamental fact that a Torelli theorem holds for $K3$ surfaces. Recall first the Torelli theorem for compact Riemann surfaces(=smooth algebraic curves). Let C be a compact Riemann surface of genus $g \geq 1$ and Ω^1_C the sheaf of germs of holomorphic 1-forms on C. Let $\omega_1, ..., \omega_g$ be a basis of $H^0(C, \Omega^1_C)$ and $\alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g$ a symplectic basis of $H_1(C, \mathbf{Z})$. Then the $g \times 2g$ matrix

$$(\int_{\alpha_j} \omega_i, \int_{\beta_j} \omega_i)_{1 \leq i,j \leq g}$$

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is called the *period matrix* of C. Roughly speaking, the Torelli theorem for compact Riemann surfaces states that the isomorphism class of C is determined by its period matrix. For $K3$ surfaces, their periods are defined by the integrals of a holomorphic 2-form over the 2nd homology classes and a Torelli-type theorem is known by Piatetskii-Shapiro and Shafarevich [18].

Definition 1.1. A surface X is called a $K3$ *surface* if
(i) X is simply connected,
(ii) there exists a nowhere vanishing holomorphic 2-form on X.

Example 1.2.
(i) Let f be a homogeneous polynomial of degree 4 such that

$$S_4 = \{(X_0 : X_1 : X_2 : X_3) \in \mathbf{P}^3 | f(X_0, X_1, X_2, X_3) = 0\}$$

is smooth. Then X is a $K3$ surface. Indeed, the simply-connectedness follows from the Lefschetz theorem for hyperplane sections, and the Poincaré residue of $dx_1 \wedge dx_2 \wedge dx_3/f(x)$ provides a nowhere vanishing holomorphic 2-form on S_4, where $x = (x_1, x_2, x_3), x = X_i/X_0$ is an affine coordinates.

The only complete intersections with canonical divisor $K \equiv 0$ are $S_4 \in \mathbf{P}^3$, $S_{2,3} \in \mathbf{P}^4$, $S_{2,2,2} \in \mathbf{P}^5$. These are $K3$ surfaces.

(ii) Let A be an abelian surface or, more generally, complex torus of dimension 2. Let τ be the involution of A given by $a \mapsto -a$. The fixed points of τ are the points of order 2 of the group A, which is isomorphic as a group to $(\mathbf{R}/\mathbf{Z})^4$; there are thus 16 points p_i of order 2. Let $\epsilon : \bar{A} \to A$ be the blow up of these 16 points; the involution τ extends to an involution σ of $\bar{A}$, fixing the exceptional curves $E_i = \epsilon^{-1}(p_i)$ pointwisely; denote by X the quotient surface $\bar{A}/\sigma$. Then X is a $K3$ surface, the *Kummer surface* of A.

2. Lattices

A *lattice* L is a free $\mathbf{Z}$-module of finite rank endowed with an integral symmetric bilinear form $\langle\ ,\ \rangle$. If L_1 and L_2 are lattices, then $L_1 \oplus L_2$ denotes the orthogonal direct sum of L_1 and L_2. Also we denote by $L^{\oplus m}$ the orthogonal direct sum of m-copies of L. An isomorphism of lattices preserving the bilinear forms is called an *isometry*. For a lattice L, we denote by $O(L)$ the group of self-isometries of L. A sublattice S of L is called *primitive* if L/S is torsion free.

A lattice L is *even* if $\langle x, x\rangle$ is even for each $x \in L$. A lattice L is *non-degenerate* if the discriminant $d(L)$ of its bilinear form is non zero, and *unimodular* if $d(L) = \pm 1$. If L is a non-degenerate lattice, the *signature* of L is a pair (t_+, t_-) where $t_\pm$ denotes the multiplicity of the eigenvalues ± 1 for the quadratic form on $L \otimes \mathbf{R}$.

Let L be a non-degenerate even lattice. The bilinear form of L determines a canonical embedding $L \subset L^* = Hom(L, \mathbf{Z})$. The factor group L^*/L, which is denoted by A_L, is an abelian group of order $|d(L)|$. We denote by $l(L)$ the number of minimal generators of A_L. We extend the bilinear form on L to the one on L^*, taking value in $\mathbf{Q}$, and define

$$q_L : A_L \to \mathbf{Q}/2\mathbf{Z},\ \ q_L(x + L) = \langle x, x\rangle + 2\mathbf{Z}\ \ (x \in L^*).$$

We call q_L the *discriminant quadratic form* of L.

We denote by U the hyperbolic lattice defined by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which is an even unimodular lattice of signature (1,1), and by A_m, D_n or E_l an even negative definite lattice associated with the Cartan matrix of type A_m, D_n or E_l ($m \geq 1$, $n \geq 4$, l = 6,7,8). A *root* of an even lattice L is a (-2)-element in L. A *root lattice* of a negative definite lattice L is the sublattice of L generated by all roots in L, which is isometric to a direct sum of A_m, D_n, E_l. For a lattice L and an integer m, $L(m)$ is the lattice whose bilinear from is the one on L multiplied by m. An even negative definite unimodular lattice of rank 24 is called *Niemeier lattice*. There are twenty-four isomorphism classes of Niemeier lattices which are uniquely determined by their root sublattice.

Next we recall a structure of the orthogonal group $O(L)$ of an even lattice L of signature $(1,r-1)$. Let $P(L)$ be a connected component of the set $\{x \in L \otimes \mathbf{R} : \langle x, x\rangle > 0\}$. Let Δ be the set of all roots in L. For each δ, an isometry s_δ defined by

$$s_\delta : x \to x + \langle x, \delta\rangle \delta$$

is called a *reflection* associated with δ. Let $W(L)^{(2)}$ be the group generated by all reflections associated with $\delta \in \Delta$. Let H_δ be the hyperplane orthogonal to $\delta \in \Delta$. Then $\{ H_\delta : \delta \in \Delta \}$ divide $P(L)$ into regions. The connected components of $P(L) \setminus (\bigcup H_\delta)$ are called the *chambers* each of which is a fundamental domain with respect to the action of $W(L)^{(2)}$ on $P(L)$. Fix a chamber $D(L)$. Then we have a partition $\Delta = \Delta^+ \cup \Delta^-$ such that Δ^+ consists of roots δ with $\langle \delta, \rho\rangle > 0$ where ρ is an interior point of $D(L)$. Let $G(L)$ be the group of symmetries of $D(L)$. Then $O(L)$ is a split extension of $\{\pm 1\} \cdot W(L)^{(2)}$ by $G(L)$. For more details we refer the reader to Vinberg [20].

3. Leech lattice

Let Ω be the projective line $PL(23)$ over the field $\mathbf{F}_{23}$: $\Omega = \{\infty, 0, 1, ..., 22\}$. We consider the set $P(\Omega)$ of all subsets of Ω with symmetric difference as a 24-dimensional vector space over $\mathbf{F}_2$. Let $\mathcal{C}$ be the binary Goray code which is a 12-dimensional subspace of $P(\Omega)$ defined by the 24 sets N_i where $N = \Omega \setminus \{x^2 : x \in \mathbf{F}_{23}\}$, $N_\infty = \Omega$ and $N_i = \{n - i : n \in N\}$ for $i \neq \infty$ (see Conway [4]). We call a set in $\mathcal{C}$ a *$\mathcal{C}$-set*. A $\mathcal{C}$-set consists of 0, 8, 12, 16 or 24 elements. A 8-elements $\mathcal{C}$-set is called an *octad*. We denote by $\mathcal{C}(8)$ the set of all octads.

The stabilizer subgroup of the binary Goray code in the permutation group of Ω is called the *Mathieu group* (denoted by M_{24}) which is a sporadic simple group of order $2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$. We denote by M_{24-k} the pointwise stabilizer subgroup of k-points in Ω.

Next we recall the Leech lattice. Let $\mathbf{R}^{24}$ be spanned by the orthogonal basis ν_i ($i \in \Omega$), and define ν_S to be $\Sigma_{i \in S} \nu_i$ for $S \subset \Omega$. Let Λ be the lattice spanned by the vectors $2\nu_K$ for $K \in \mathcal{C}(8)$ and $\nu_\Omega - 4\nu_\infty$.

For $x, y \in \Lambda$, define $\langle x, y\rangle = -x \cdot y/8$. Then $(\Lambda, \langle , \rangle)$ is an even negative-definite unimodular lattice which is called the *Leech lattice*.

Finally we recall a structure of the orthogonal group of even unimodular lattice of signature (1,25). Write L for an even unimodular lattice of signature (1,25), which is unique up to isomorphisms and thus isomorphic to $\Lambda \oplus U$. Fix a decomposition $L = \Lambda \oplus U$. We write (λ, m, n) for a vector in L where λ in Λ, m, n are integers, the norm is given by $2mn + \langle \lambda, \lambda\rangle$. If $w = (0, 0, 1)$, then a root r in L with $\langle r, w\rangle = 1$ is called a *Leech root* and w is called a *Weyl vector*. The set of all Leech roots bijectively corresponds to the set Λ as follows (Conway-Sloane [6], Chap. 26, Theorem 3):

$$L \ni r = (\lambda, 1, -1 - \langle\lambda, \lambda\rangle/2) \longleftrightarrow \lambda \in \Lambda.$$

Put

$$D = \{x \in P(L) \mid \langle x, r\rangle \geq 0 \text{ for any Leech root } r\}.$$

Proposition 3.5 (Conway [5]). *D is a fundamental domain of the reflection group $W(L)^{(2)}$ and $O(L)$ is a split extension of $W(L)^{(2)}$ by $Aut(D)$ so that $Aut(D)$ is isomorphic to the group $\cdot\infty$ of affine automorphisms of Λ, which is a split extension of $\mathbf{Z}^{24}$ by $\cdot 0 = O(\Lambda)$*

4. Periods of $K3$ surfaces

Let X be a $K3$ surface. Then it follows from Definition 1.1 and Noether formula that the first Chern class $c_1(X) = 0$ and the Euler number $e(X) = 24$. Since X is simply-connected, the Betti numbers are : $b_1(X) = b_3(X) = 0$, $b_2(X) = 22$. Moreover, $H^2(X, \mathbf{Z})$ is a free $\mathbf{Z}$-module with a lattice structure induced from the cup product $\langle,\rangle$. This lattice is of signature (3,19) (Hirzebruch signature theorem), even (Wu's formula), and unimodular (by Poincaré duality). Therefore by a theorem of Milnor, we have a lattice isomorphism

$$H^2(X, \mathbf{Z}) \cong U \oplus U \oplus U \oplus E_8 \oplus E_8.$$

Definition 4.1. Let ω be a nowhere vanishing holomorphic 2-form on a $K3$ surface X. Then ω is a basis of $H^0(X, \Omega_X^2)$ where Ω_X^2 is a sheaf of germs of holomorphic 2-forms on X. By Dolbeault's theorem, we can identify $H^0(X, \Omega_X^2)$ with a 1-dimensional subspace of $H^2(X, \mathbf{C})$. Now we define

$$S_X = \{x \in H^2(X, \mathbf{Z}) | \langle x, \omega\rangle = 0\},$$
$$T_X = \{x \in H^2(X, \mathbf{Z}) | \langle x, y\rangle = 0 \quad \text{for} \quad \text{all} \quad y \in S_X\}.$$

Both S_X and T_X are primitive sublattices of $H^2(X, \mathbf{Z})$. We call S_X the *Picard lattice* of X (it is isomorphic to the divisor class group $H^1(X, \mathcal{O}_X^*)$) and T_X the *transcendental lattice* of X. We remark that $H^0(X, \Omega_X^2) \subset T_X \otimes \mathbf{C}$ and $A_{S_X} \cong A_{T_X}$ (since $H^2(X, \mathbf{Z})$ is unimodular).

Let $H^2(X, \mathbf{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$ be the Hodge decomposition and put

$$H^{1,1}(X)_{\mathbf{R}} = H^{1,1}(X) \cap H^2(X, \mathbf{R}).$$

Then the signature of the cup product on $H^{1,1}(X)_{\mathbf{R}}$ is (1,19), and hence the cone

$$\{x \in H^{1,1}(X)_{\mathbf{R}} | \langle x, x\rangle > 0\}$$

has two connected components. We denote by $P(X)$ the component containing a Kähler class(=a class of (1,1)-form associated to a Kähler metric). Put

$$D(X) = \{x \in P(X) | \langle x, \delta\rangle > 0 \quad \text{for} \quad \text{all} \quad \text{effective} \quad \text{divisor} \quad \delta \in S_X, \langle\delta, \delta\rangle = -2\}.$$

We call $D(X)$ the *Kähler cone* of X. As its name suggests, $D(X)$ consists of all Kähler classes.

Theorem 4.2(Piatetskii-Shapiro and Shafarevich[18], Burns and Ropoport[3]). *Let X and X' be $K3$ surfaces and let*

$$\psi : H^2(X, \mathbf{Z}) \to H^2(X', \mathbf{Z})$$

be a lattice isometry. Then there exists an isomorphism $\phi : X' \to X$ with $\phi^* = \psi$ if and only if

(i) $(\psi \otimes \mathbf{C})(H^0(X, \Omega_X^2)) = H^0(X', {\Omega'_X}^2)$,

(ii) $\psi \otimes \mathbf{R}$ *preserves Kähler cones.*

Moreover, ϕ is unique if it exists.

5. Automorphisms of $K3$ surfaces

Let $\mathrm{Aut}(X)$ denote the group of (biholomorphic) automorphisms of a $K3$ surface X. Then it follows from the Torelli theorem (4.2)(together with some lattice theory) that $\mathrm{Aut}(X)$ is isomorphic, up to finite groups, to the group of isometries of the Picard lattice S_X which preserve the Kähler cone $D(X)$ of X. In other words

$$\mathrm{Aut}(X) \cong O(S_X)/W(S_X)^{(2)},$$

up to finite groups. In particular,

$$|O(S_X)/W(S_X)^{(2)}| < \infty \Longleftrightarrow |\mathrm{Aut}(X)| < \infty$$

($\Longleftrightarrow$ X contains only finitely many smooth rational curves).

In this case, Nikulin[14] classified all such lattices S_X. It follows from his classification that every algebraic Kummer surface has an infinite group of automorphisms(the Picard lattice of no Kummer surface can belong to his list).

In the case where $\mathrm{Aut}(X)$ is infinite, it is in general difficult to describe $\mathrm{Aut}(X)$ explicitly. In the following we will mention a few examples whose automorphism groups are fully known.

(1) *Two most algebraic $K3$ surfaces*

Vinberg[21] calculated $\mathrm{Aut}(X)$ for two $K3$ surface with transcendental lattice $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ respectively. In both cases, the full reflection group $W(S_X)$ is of finite index in $O(S_X)$.

(2) *generic Jacobian Kummer surfaces*

Let C be a smooth curve of genus 2. The Jacobian variety $J(C)$ of C is an abelian surface with a natural involution τ and the quotient variety $J(C)/\tau$ has 16 ordinary nodes. This surface can be embedded as a quartic surface F in $\mathbf{P}^3$ with 16 nodes. The minimal resolution X of $J(C)/\tau$ is called the *Jacobian Kummer surface* associated with C. We call X *generic* if the Néron-Severi group of $J(C)$ is generated by the class of C.

At the last century it was known that X has many involutions, that is, *sixteen translations* induced by those of $J(C)$ by a 2-torsion point, *sixteen projections* of F from a node, *sixteen correlations* by means of the tangent plane collinear to a trope, and a *switch* defined by the dual map of F. In 1900, Hutchinson found another 60 involutions associated with Göpel tetrads. Since Hutchinson, for generic X no other automorphism was provided until the author found new 192 automorphisms.

Theorem 5.1(Keum[7]). *For a generic Jacobian Kummer surface, there are* 192 *new automorphisms of infinite order which are not generated by classical involutions.*

Theorem 5.2(Kondo[11]). *The automorphism group of a generic Jacobian Kummer surface is generated by classical involutions and the* 192 *new automorphisms.*

Theorem 5.3(Keum[8]). *For F generic, all birational automorphisms of F are induced by Cremona transformations(=birational transformations of* $\mathbf{P}^3$*).*

(3) *Kummer surfaces associated with the product of two elliptic curves*

Recently, Keum and Kondo[9] have calculated the automorphism groups of the following Kummer surfaces X:

Case I. $X = Km(E \times F)$ where E and F are non-isogenus generic elliptic curves. In this case $T_X = U(2) \oplus U(2)$.

Case II. $X = Km(E \times E)$ where E is an elliptic curve without complex multiplications. In this case $T_X = U(2) \oplus < 4 >$.

Case III. $X = Km(E_\omega \times E_\omega)$ where ω is a 3rd root of unity and E_τ is the elliptic curve with τ as its fundamental period. In this case $T_X = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$.

Case IV. $X = Km(E_{\sqrt{-1}} \times E_{\sqrt{-1}})$. In this case $T_X = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

The outline of the proof which is similar to that of Kondo for a generic Jacobian Kummer surface goes as follows. Recall that Kummer surfaces are $K3$ surfaces. It follows from Piatetski-Shapiro, Shafarevich that the group of automorphisms is isomorphic, up to finite groups, to the group of isometries of the Picard lattice S_X of X which preserve the Kähler cone $D(X)$ of X. In above cases, $Aut(X)$ is infinite and hence it is difficult to describe the Kähler cone explicitly. On the other hand, by using results of Conway and Borcherds, we are able to find a polyhedral cone D' in the Kähler cone $D(X)$ of X bounded by finite number of faces. Next for each face of D' we constructed an automorphism of X which works like a reflection with respect to this face. A standard argument shows that $Aut(D')$ and these automorphisms generate a subgroup of finite index in the orthogonal group $O(S_X)$. To sum up, the proof consists of the following three major steps:

Step 1. Find a primitive embedding of Picard lattice S_X of the surface X into the extended Leech lattice L of signature $(1, 25)$.

Step 2. Use the result of Conway and Borcherds on the fundamental domain $D(L)$ of L to find all faces of the polyhedral cone D'. D' is nothing but the intersection of $D(L)$ with $P(S_X) \subset P(L)$

Step 3. For each face of D', construct an automorphism of X which works like a reflection with respect to the face. For this, we used elliptic fibration structure of X.

This method can be applied to some other $K3$ surfaces, e.g. Fermat quartic surface, $X_0^4 + X_1^4 + X_2^4 + X_3^4 = 0$ (Keum-Kondo[10]).

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