Trends in Mathematics
Information Center for Mathematical Sciences
Volume 1, Number 1, September 1998, Pages 11–18

GEOMETRIC COMPACTIFICATION OF MODULI OF SURFACES OF GENERAL TYPE

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ABSTRACT. This short article is an introductory surveying of a recent work of the construction of geometric compactification of moduli of surfaces of general type by using higher dimensional geometry. We introduce how to obtain boundary surfaces and discuss the bounds problem of compactified moduli of surfaces. And we provide some related problems in this direction.

0. INTRODUCTION

We work throughout over the complex number field $\mathbb{C}$. The notation here follows the standard text [Hartshorne's algebraic geometry].

It has been known for a long time the set of all curves of a fixed genus g naturally corresponds to the points on a certain variety with dimension $3g-3$, which is denoted by $\mathcal{M}_g$. The moduli space $\mathcal{M}_g$ is not proper, and there has been considerable work in finding a natural compactification. What kind of curves correspond to the boundary of $\mathcal{M}_g$? An answer was provided by Deligne and Mumford in [DM]. This compactification is called Deligne-Mumford compactification of $\mathcal{M}_g$, denoted by $\overline{\mathcal{M}_g}$. A stable curve of genus g is defined as a 1-dimensional proper scheme D which has only ordinary nodes as singularities such that ω_D is ample and $\chi(D, \mathcal{O}_D) = 1-g$.

Gieseker [G] proved the existence of a quasi projective coarse moduli space $\mathcal{M}(X)$ for smooth projective surface X of general type with fixed numerical invariants $\chi(\mathcal{O}_X)$ and K_X^2. The next natural problem is how to construct a geometric compactification of moduli of surfaces via similar way as $\overline{\mathcal{M}_g}$: Kollár and Shepherd-Barron [KSB] found a geometric approach to this problem using the minimal model program in higher dimensional geometry. The compactified moduli space should include (possibly reducible) surfaces with ordinary double curves and certain other mild singularities. The frame work of this compactified moduli of surfaces of general type was done by Alexeev's proof [A] of bounds for log surfaces with given K^2. Recently Karu in [Karu] give another proof by using minimal model program, weakly semistable reduction theorem and invariance of plurigenera. But it is still necessary to complete many detailed works and to provide explicit examples. The notion of discrepancy is the fundamental measure of the singularities of (X, D) (cf. [CKM], [K et] or [KM]).

Definition. Let X be a normal variety and $D = \sum d_i D_i$ an effective $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. Let $f: Y \to X$ be a proper birational morphism.

1991 *Mathematics Subject Classification.* 14J10, 14J29.
Key words and phrases. log canonical singularity, stable surface, index of singularity.

Numerical equivalence of divisors or line bundles is denoted by $\equiv$. Then we can write

$$K_Y + f_*^{-1}(D) \equiv f^*(K_X + D) + \sum a(E, X, D)E$$

where $f_*^{-1}(D)$ is the proper transform of D, the sum runs over distinct prime divisors $E \subset Y$, and $a(E, X, D) \in \mathbb{Q}$. This $a(E, X, D)$ is called the *discrepancy* of E with respect to (X, D); in fact it only depends on the divisor E, and not on the partial resolution Y.

Let X be a normal variety and $D = \sum d_i D_i$ an effective $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. We define

$$\mathrm{discrep}(X, D) = \inf_E \{a(E, X, D) | E \text{ is exceptional, and } \mathrm{Center}_X(E) \neq \emptyset\}$$

We say that (X, D), or $K_X + D$ is

$$\left.\begin{array}{l}\text{terminal}\\ \text{canonical}\\ \text{purely log terminal}\\ \text{log canonical}\end{array}\right\} \quad \text{if} \quad \mathrm{discrep}(X, D) \quad \left\{\begin{array}{l} > 0,\\ \geq 0,\\ > -1,\\ \geq -1.\end{array}\right.$$

Moreover, (X, D) is *Kawamata log terminal* (klt) if (X, D) is purely log terminal, $d_i < 1$ for every i; and (X, D) is *divisorial log terminal* (dlt) if there exists a log resolution so that exceptional locus consists of divisors with all $a(E, X, D) > -1$.

Assume that X is a normal surface, and $D = \sum d_i D_i$ with all $d_i = 1$, $D \neq 0$. We draw the graph of curves on the minimal resolution of X using the following convention: $\bullet$ represents a component of D and $\circ$ represents an exceptional curve of a resolution. Then by [K et] or [KM], it is known that (X, D) is dlt if and only if

1. X is smooth and D of the form $\bullet - \bullet$, or
2. X has cyclic quotient singularity and total transform of D of the form $\bullet - \circ - \cdots - \circ$

In section 1, we introduce how to construct a stable surface and we provide some questions in this direction. In section 2, we discuss some bounds problems of compactified moduli of surfaces. And in section 3, we discuss and give some questions for construction of explicit examples.

1. Construction of a stable surface

1.1 Let $0 \in T$ be the Spec of a DVR (or unit disc) and assume that we have a family of surfaces of general type $f^0 : \mathcal{X}^0 \to (T - 0)$. This family is assumed minimal (K_{X_t} is nef for all $t \in (T - 0)$). By log resolution of singularities [H] and semi-stable reduction theorem [KKMS], there is a ramified base change $T^{'} \to T$ such that $T^{'} \times_T \mathcal{X}^0$ can be extended to $f^1 : \mathcal{X}^{'} \to T^{'}$. And $\mathcal{X}^{'}$ is smooth and the central fiber of f^1 is reduced with normal crossings only. This $f^1 : \mathcal{X}^{'} \to T^{'}$ will be called a semi-stable degeneration of surfaces. Note that $(\mathcal{X}^{'} - X_0^{'}) \to (\mathcal{X} - X_0)$ is etale. Since log resolution of singularities and semi-stable reduction theorem work for any dimension, one can obtain a semi-stable degeneration for any dimension. It is interesting problem to construct a semi-stable degeneration without using a special resolution of toric singularities (resolution of toric singularities is a main part of semi-stable reduction theorem). After log resolution of singularities of one-parameter family of curves, it is enough to change base and to normalize for constructing a semi-stable degeneration of curves [BPV, III. 9, 10]. Kollár and Mori [KM] obtained the semi-stable reduction theorem for degeneration of surfaces via terminalization and $\mathbb{Q}$-factorialization instead of special resolution of toric singularities.

If each component of the central fiber of a semi-stable degeneration satisfies the Kähler condition, then one can find some relations between cohomologies of general fiber, and cohomlogies of components of central fiber, of double divisors (double divisor is the divisor of a component coming from the intersection of other components), of dual graph of the central fiber, via using the Clemens-Schmid exact sequence that relate topology and mixed Hodge theory of the central fiber $X_0^{'}$ to that of general fiber $X_t^{'}$ through the monodromy of the restricted family $(\mathcal{X}^{'} - X_0^{'}) \to (T^{'} - 0)$. The proofs, or details of the statements of Clemens-Schmid exact sequence can be found in references [Cl], [EMS, Ch.4, 5], [Mo] or [P]. Therefore a semi-stable degeneration has a good property for the restriction of the central fiber, but one can add rational or ruled surfaces to the central fiber by simply blowing-up at points or smooth curves.

The following question can be natural: Is it possible to find full classification (up to rational or ruled surfaces) of a central fiber of a semi-stable degeneration of surfaces with given K^2, $\chi(\mathcal{O}_X)$? How is it about the case: $K^2 = 1$ $\chi(\mathcal{O}_X) = 1$ (numerical Godeaux surface)? Does the classification of the central fiber help to understand the moduli space of numerical Godeaux surfaces which is unknown yet? Conversely the following question can be asked: Under a suitable conditions, is there a global smoothing of a normal crossing surface X_0? Global smoothing was studied by Friedman in [F], but almost no information is known for the case ω_{X_0} being ample. For a semi-stable degeneration of $K3$-surfaces, the full classification of a central fiber and global smoothings are known [F] or [EMS, Ch.6].

1.2 Let $f^1 \colon \mathcal{X}^{'} \to T^{'}$ be a semi-stable degeneration of surfaces of general type. In particular, $\mathcal{X}^{'} \to T^{'}$ satisfies the following:

1. $\mathcal{X}^{'}$ has terminal singularities.
2. $X_0^{'}$ is a reduced Cartier divisor and is $\equiv 0$ relative to f^1.
3. $f^1 \colon \mathcal{X}^{'} \to T^{'}$ is dlt $((\mathcal{X}^{'}, (f^1)^{-1}(t))$ is dlt for all $t \in T^{'})$.
4. $\mathcal{X}^{'}$ is $\mathbb{Q}$-factorial.

Then, by [KM], running the $K_{\mathcal{X}^{'}/T^{'}}$-MMP (which is the same thing as running the $K_{\mathcal{X}^{'}/T^{'}} + X_0^{'}$-MMP) preserves properties (1)–(4). So we obtain a new family $f^2 \colon \mathcal{X}^{''} \to T^{''}$ satisfying (1)–(4) and

5. $K_{\mathcal{X}^{''}/T^{''}}$ is relatively nef.

This semi-stable minimal model program is much simpler than the general case [KM]. Let (V, D_V) be a pair of a component and a double curve on the central fiber of a relatively minimal model. Then (V, D_V) has dlt singularities. This relatively minimal degeneration is not unique but there are only finitely many possibilities [Ka1].

1.3 Apply relative base point freeness to get a $f^3 \colon \mathcal{X}^3 \to T^3$ where the canonical class is relatively ample. Then $f^3 \colon \mathcal{X}^3 \to T^3$ satisfies the following:

1. $\mathcal{X}^3$ has canonical singularities.
2. X_0^3 is a reduced Cartier divisor and is $\equiv 0$ relative to f^3.
3. $f^3 \colon \mathcal{X}^3 \to T^3$ is log canonical $((\mathcal{X}^3, (f^3)^{-1}(t))$ is log canonical for all $t \in T^3)$.
4. $\mathcal{X}^3$ may not be $\mathbb{Q}$-factorial but $K_{\mathcal{X}^3/T^3}$ is $\mathbb{Q}$-Cartier.

5. $K_{\mathcal{X}^3/T^3}$ is relatively ample.

This central fiber is called a **stable surface**. Let (V, D_V) be a pair of a component and a double curve in the central fiber. Then the normalization of (V, D_V) has log canonical singularities ((V, D_V) has semi-log canonical singularities).

Let $g\colon \mathcal{X} \to C$ be a morphism from a surface to a smooth curve whose generic fiber is smooth. Then the following are equivalent:

1. Every fiber has at most ordinary nodes as singularities.
2. If $C^{'}$ is any smooth curve and $h\colon C^{'} \to C$ is a nonconstant morphism then $\mathcal{X} \times_C C^{'}$ is canonical.
3. $g\colon \mathcal{X} \to C$ is log canonical.

But the above equivalence is not true for a higher dimensional variety. Log canonical morphisms can not be characterized by their fibers. Consider the following example [KM]: Consider the Veronese surface $V \subset \mathbb{P}^5$ and the degree 4 rational normal scroll $S \subset \mathbb{P}^5$ (embedding F_2 by $C_0 + 3f$). Both surfaces have the degree 4 rational normal curve $D \subset \mathbb{P}^4$ as their general hyperplane section. Then we can construct 2 one-parameter family of surfaces $g^1\colon \mathcal{X}^1 \to T^1$ and $g^2\colon \mathcal{X}^2 \to T^2$ such that $(g^1)^{-1}(t) = V$, $(g^2)^{-1}(t) = S$ for $t \in (T^1 - 0)$ or $t \in (T^2 - 0)$ and $(g^1)^{-1}(0) = (g^2)^{-1}(0) = C_D$ where $C_D \subset \mathbb{P}^6$ is the cone over D. The minimal resolution of C_D is F_4. Note that each fiber of 2 one-parameter family of surfaces has log canonical singularities but $(K_V^2) = (K_{C_D}^2) = 9$, $(K_S^2) = 8$. This example shows that for family of surfaces $g\colon \mathcal{X} \to C$ over a smooth curve, we need to assume that $K_{\mathcal{X}/C}$ is $\mathbb{Q}$-Cartier. For families over a general base, this condition becomes more subtle. A stable surface X is defined as a semi-log canonical surface occured as the central fiber of a smoothing $\mathcal{X} \to T$ which has the above five properties.

Stable curves are asymptotically Hilbert-stable [Mu], then the GIT theory produces a quasi-projective moduli space. Since it is proper, it is a projective scheme. By [Mu] in order to be asymptotically Chow- or Hilbert-stable surface has to have singularities of multiplicities at most 7. However a stable surface of our construction has arbitrarily high multiplicities, it is mentioned in [Ko2] and [SB]. So the approach by GIT fails for surfaces of general type.

2. Bounds problem of compactified moduli space

2.1 To construct the moduli functor $\mathcal{M}$ of surfaces of general type with given $K^2 = A$ and $\chi(\mathcal{O}_X) = B$, which is coarsely represented projective scheme, the following should be answered [KSB].

1. Does there exist an integer N depending only on A and B, and whenever X is a stable surface with $K_X^2 = A$ and $\chi(\mathcal{O}_X) = B$, the index of singularities of X divides N?
2. Does there exist an integer M such that every stable surface X with $K_X^2 = A$ and $\chi(\mathcal{O}_X) = B$ whose index divides N, the sheaf $(\omega_X)^{NM}$ is very ample?
3. Is Aut X finite for a stable surface X?
4. If $\mathcal{X} \to T$ is a one-parameter family of surfaces such that the special fiber is a stable surface with $\omega_{\mathcal{X}/T}$ being $\mathbb{Q}$-Cartier, then is the generic fiber a stable surface?

The second one is answered by Kollár in [Ko1]. The third by Itaka in [I], and the fourth by Kollár and Shepherd-Barron in [KSB]. The first one is called the bounds problem, which is theoretically done by Alexeev in [A].

2.2 Alexeev and Mori [AM] subsequently found simpler proofs of these bounds, and their theorems also contain explicit bounds; but these are usually unrealistically high, and explicit bounds for the index of singularities are not known. In particular, to construct explicit examples it is necessary to find explicit bounds and to get more clear picture of the singularities on a stable surface. Given A and B, we [L2] obtain that the number of components in a stable surface is bounded by $12B - A$. There are some examples which make this bounds to be sharp. We use Kawamata's topological arguments in [Ka2] for a permissible degeneration (this degeneration is a small modification of a general relatively minimal degeneration for simplifying singularities), and Wahl's generalization [W] of Miyaoka's theorem [Mi2] bounding the number of quotient singularities on surfaces with given numerical invariants. Via studying the resolution graph of quotient singularities coming from stable surfaces, we [L2] obtain a bound for an index of singularities of a special stable surface which has only one singularity. It is bounded by 2^{400A^4}. This number is quite high, but it illustrates the reason behind the bounds for the index in a stable surface. For a surface of general type, there is a Bogomolov-Miyaoka-Yau inequality [Mi1], which is one of the fundamental inequality. In some sense, many boundness problems of log surface are related to the extension problem of this inequality to the log surface. Some of extension was done by Miyaoka [Mi2], Wahl [W] and Megyesi [K et, Ch. 10]. It is hoped that further extension is possible and it provides some answers to the explicit bounds for a compactifying moduli space. There are some conjectures for explicit bounds for log surfaces in [Ko3].

3. Complete moduli space for fibered surfaces and explicit examples

3.1 Recently Abramovich and Vistoli [AV] construct completing a moduli space for fibered surfaces via using the compactifying the space of stable maps, and they compared their work with Alexeev's work. The Abramovich-Vistoli method for completing moduli for fibered surfaces works for constructing some explicit examples. The moduli space of surfaces of general type with $p_g = q = a$, $K^2 = b$ studied by Catanese and Ciliberto [Ca1] [CC]. In particular the case $p_g = q = a$, $K^2 = 8$ is our main interest. From our theorem in [L2], the number of components in a stable surface is bounded by Euler number, and this particular case is the second smallest possible for surfaces of general type. (The smallest case is a ball quotient, so the moduli space is finite). It is interesting to describe the moduli space $\mathcal{M}_{\chi(\mathcal{O}_X)=1,K^2=8}$ and to describe its stable in terms of a stable degeneration of the base curve and the fibers of a multicanonical pencil. In general, we need to blow up at some base points of a linear system to construct a fibered surface. For example, one can construct a fibered surface after blowing up the four base points of the bicanonical pencil of numerical Godeaux surface. An approach to describing the stable surfaces in these cases is to complete moduli space for the associated fibered surfaces and then to blow down exceptional parts. One reason to understand well the behavior of the boundary in a compactified moduli space of surfaces of general type is the following. By Manetti's recent work in [Ma], some irreducible components in $\mathcal{M}_{\chi(\mathcal{O}_X)=a,K^2=b}$ can be connected by stable surfaces on the boundary of different components of the moduli space. In [L1], we present a geometric description of the compactification of the family of determinantal Godeaux surfaces, via the study of the bicanonical pencil and using classical Prym theory. In this work, we tried to reduce the problem of compactifying the special family of surfaces to the problem of compactifying the pencils and the family of curves over the pencils.

3.2 In general, the understanding of the moduli space of surfaces with given numerical invariants requires constructing a geometrically meaningful compactification. But such a compactification is still unknown even for surfaces of general type with small numerical invariants. In a degeneration of surfaces of general type with large numerical invariants, it is quite possible that surfaces of small numerical invariants occur as a minimal resolution of components of stable surfaces at the limit. Is it possible full description of a nontrivial stable surface (stable surface with non DuVal Singularity) without knowing a moduli space of surfaces with given numerical invariants? For example, can one describe a nontrivial stable surfaces of numerical Godeaux surfaces even though the moduli space is unknown? A numerical Godeaux surface is a surface of general type with $p_g = q = 0$ and $K^2 = 1$. There are five families of numerical Godeaux surfaces distinguished by $Tor\, X$, which may take the values $0, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_5$ [R]. The dimension and irreducibility of the moduli of numerical Godeaux surfaces with $Tor\, X = 0$ is unknown yet. The first example of this kind of numerical Godeaux surface was constructed by Barlow in [B], recently another example is constructed by Craighero-Gattazzo-Dolgachev-Werner in [DW]. The simplest example of a numerical Godeaux surface is constructed as a quotient of a quintic in $\mathbb{P}^3$ under a free $\mathbb{Z}_5$-action, it is called a Godeaux surface. For the generic such Godeaux surface X, it is easily computed that $h^1(T_X) = 8$ and $h^2(T_X) = 0$. In the 8-dimensional family of Godeaux surfaces, there is a four-dimensional subfamily for which the quintic is symmetric determinantal. This family was studied by Catanese. We call these surfaces determinantal Godeaux surfaces. Again, inside this four-dimensional family, there is a two-dimensional subfamily for which the $\mathbb{Z}_5$-action on the (symmetric determinantal) quintic extends to an action of the dihedral group D_5. By twisting this action, Barlow constructed the first example of a simply connected surface of general type with $p_g = 0$. In fact, as Barlow remarks, her twisting works for the entire two dimensional family mentioned above, giving a two dimensional family of simply connected surfaces which we call determinantal Barlow surfaces. In [L3], we study the bicanonical pencil of a Godeaux surface and of a determinantal Barlow surface. This study gives a simple proof for the unobstructedness of deformations of a determinantal Barlow surface. Then we compute the number of hyperelliptic curves in the bicanonical pencil of a determinantal Barlow surface via classical Prym theory. It is not known at present whether Craighero-Gattazzo-Dolgachev-Werner's example in [DW] is deformation equivalent to Barlow's example. It is possible to understand Craighero-Gattazzo-Dolgachev-Werner's example by our bicanonical method in [L1] and [L3]. Under some strong conditions, we can classify a nontrivial stable surfaces of numerical Godeaux surfaces. For a numerical Godeaux surface, minimal model of each component of nontrivial stable surface is not general type. Therefore it may be possible to classify nontrivial stable surfaces via using the classification theory of surfaces of non-general type. Given A and B, the moduli space of surfaces of general type $\mathcal{M}_{A,B}$ with $K^2 = A$, $\chi(\mathcal{O}_X) = B$ is very little known. Some explicit bounds were proved by Catanese [Ca2], [Ca3] (dimension of $\mathcal{M}_{A,B} \leq 10B + 3A + 108$, the number of irreducible components of $\mathcal{M}_{A,B} \leq 6^{(A+5/9)^{15}}$ if $A \geq 3$). It is also very interesting problem to study $\mathcal{M}_{A,B}$.

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