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CONJECTURES ON COHOMOLOGY VANISHING OF AN IDEAL SHEAF OF A PROJECTIVE VARIETY AND ITS DEFINING EQUATIONS

SIJONG KWAK

Abstract. The main goal of this survey note is to introduce some interesting and long standing conjectures on cohomology vanishing of an ideal sheaf of a projective algebraic variety and its defining equations to mathematicians with modest background on algebraic geometry. In particular, these conjectures involve the notion of higher order normality, which is useful to distinguish between general and special behaviors of such varieties and their equations. To achieve algebraic understandings and geometric interpretations of these conjectures, it is suggested to undertake a thorough investigation of how the geometry of projective space is reflected in the geometry of its algebraic subvarieties, particularly, those of small codimension and to understand vector bundle technics involved in those conjectures. The tools to be used include multisecant lines, higher secant varieties, generic projections, finite schemes, free resolution of the sheaf of ideals of a given variety, vector bundles over a projective space defined over complex numbers.

General Background and some Conjectures

Let $X^n \subset \mathbb{P}^N$ be a nondegenerate projective variety of dimension n and degree d over $\mathbb{C}$, defined by the sheaf of ideals $\mathcal{I}_X \subset \mathcal{O}_{\mathbb{P}^N}$ and let $I_X = \bigoplus_{m\in\mathbb{Z}} H^0(\mathbb{P}^N, \mathcal{I}_X(m))$ $= (F_1, F_2, \ldots, F_m)$, $m \geq N-n$ be the saturated ideal of X which is finitely generated and we call the F_i the defining equations of X. Of course, X is an (set theoretically and scheme theoretically) intersection of hypersurfaces F_i. The degree of X is defined as the number of distinct points in $X^n \cap \Lambda^{N-n}$ for a general linear subspace Λ of dimension $N-n$ in $\mathbb{P}^N$. We can also define the affine cone of X, denoted by $\mathcal{C}(X)$ as $\mathrm{Spec}\mathbb{C}[T_0, T_1, \ldots, T_N]/I_X$, i.e., it is the zero locus of same equations in an affine space $\mathbb{A}^{N+1}$.

In general, one can easily show that a variety X of degree d is an intersection of hypersurfaces (actually cones) of degree at most d set-theoretically. Furthermore, if X is smooth, then X is scheme-theoretically cut out by homogeneous polynomials of degree d, i.e., there is a surjection $\bigoplus_{i=1}^m \mathcal{O}_{\mathbb{P}^N}(-d) \to \mathcal{I}_X$ with $m \geq N-n$, [Mu2]. X is said to be k-normal if the homomorphism

$$H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}}(k)) \to H^0(X, \mathcal{O}_X(k))$$

is surjective, i.e., hypersurfaces of degree k cut out a complete linear system on X, equivalently the cohomology group $H^1(\mathbb{P}^N, \mathcal{I}_X(k)) = 0$. We know by Serre's vanishing theorem ([Ha 2], III Theorem 5.2.) that X is k-normal for all $k >> 0$.

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It would be very interesting to find an *explicit* bound k_0 in terms of its invariants such that X is k-normal for all $k \geq k_0$ for all nondegenerate projective varieties.

The most basic projective invariants of the variety X are given by the cohomology groups $H^i(\mathbb{P}^N, \mathcal{I}_X(j))$, $i, j \geq 0$. The group $H^0(\mathbb{P}^N, \mathcal{I}_X(j))$ measures the number of equations of X of degree j, the group $H^1(\mathbb{P}^N, \mathcal{I}_X(j))$ measures the failure of the linear system on X cut out by hypersurfaces of degree j to be complete, and since $H^i(\mathbb{P}^N, \mathcal{I}_X(j)) \simeq H^{i-1}(X, \mathcal{O}_X(j))$ for $i \geq 2$, the higher cohomology groups measure "irregularity" or "specialty" of the variety X.

The simplest projective algebraic variety is a *complete intersection* of hypersurfaces, i.e., the number of minimal generators of the saturated ideal I_X is equal to its codimension. Complete intersections form a very small subclass of all varieties; though we can not prove here, there are numerous restrictions on the topology of complete intersections and on their numerical invariants (e.g., the cohomology groups $H^i(\mathbb{P}^N, \mathcal{I}_X(j)) = 0$ for $1 \leq i \leq \dim(X)$, $j \in \mathbb{Z}$ and if they have dimension two or more, they must be simply connected).

The second simplest variety is a *projectively Cohen-Macaulay* variety, i.e., by definition the local ring of its affine cone at the vertex $P = (0, 0, \cdots, 0) \in \mathcal{C}(X)$, $\mathcal{O}_{\mathcal{C}(X),P}$ is Cohen-Macaulay which is equivalent to the cohomology groups $H^i(\mathbb{P}^N, \mathcal{I}_X(j)) = 0$ for $1 \leq i \leq \dim(X)$, $j \in \mathbb{Z}$. On the other hand, X is *projectively normal* if (by definition) the local ring at the vertex P, $\mathcal{O}_{\mathcal{C}(X),P}$ is integrally closed which is equivalent to the cohomology groups $H^1(\mathbb{P}^N, \mathcal{I}_X(j)) = 0$ for $j \in \mathbb{Z}$ and X is normal. (These equivalences can be proved by using the relation between local cohomology and global cohomology, see [Ei].)

Hartshorne's conjecture. *For a smooth variety X of dimension n in $\mathbb{P}^N$, X is a complete intersection if $N < \frac{3}{2}n$*

This conjecture was proposed for the first time by R. Hartshorne [Ha1] in 1974, and it is still open even the case of smooth projective varieties of codimension two. For a smooth codimension two variety X of $\dim(X) \geq 4$, by Serre's construction it is defined scheme-theoretically as the zero locus of a section of a unique vector bundle of rank two (corresponding to X) over its ambient projective space. (see [OSS], Theorem 5.1.1.). In this case, it is also known that X is a complete intersection if and only if its corresponding rank two vector bundle splits as a sum of trivial line bundles (see [OSS], Lemma 5.2.1.) if and only if it is projectively normal (due to Gherardelli, Gaeta, Peskine, and Grifiths and Evans) and in particular, by Hartshorne's conjecture $X^{N-2} \subset \mathbb{P}^N$ should be a complete intersection for $N \geq 6$ [H1]. There is no known smooth projective variety of codimension two other than complete intersections if $\dim(X) \geq 4$. In addition, there is only one known non-splitting rank two vector bundle over a projective space defined over the complex numbers, that is, the Horrock-Mumford bundle over $\mathbb{P}^4$.

On the other hand, it is observed that for $i \geq 1$, the cohomology groups $H^i(\mathcal{I}_X(j))$ tend to vanish for varieties of small codimension. Recently, the case of smooth varieties of codimension two was studied to show the vanishing of $H^i(\mathcal{I}_X(j))$ in several papers. In other words, many mathematicians would like to show that X in the Hartshorne's range looks numerically like a complete intersection.

In [Ran3], it is proved $H^1(\mathbb{P}^N, \mathcal{I}_X(j)) = 0$ for $j \geq j(3j+2)$. Then Peternell, Le Potier and Schneider showed [PPS] that $H^1(\mathbb{P}^N, \mathcal{I}_X(2)) = 0$ for $\dim(X) \geq 10$. This result was improved by Ein who established similar vanishing for $\dim(X) \geq 8$ and proved that $H^1(\mathbb{P}^N, \mathcal{I}_X(j)) = 0$ for $\dim(X) \geq j + 10$ [E2]. Alzati and Ottaviani published a series of papers on this subject which culminated in [AO] where they got the linear bound on the t-normality of codimension two smooth projective varieties of $\dim(X) \geq 4$, i.e., $H^1(\mathbb{P}^N, \mathcal{I}_X(t)) = 0$ for $N \geq 6t - 2 (t \geq 2)$ and $H^i(\mathbb{P}^N, \mathcal{I}_X(t)) = 0$ for $i \geq 1, N \geq 6t + i$.

For higher cohomology groups, a Lefschetz type theorem says that $H^i(\mathbb{P}^N, \mathcal{I}_X) \simeq H^{i-1}(X, \mathcal{O}_X) = 0$ for $i \leq 2n - N + 1$ [B]. Furthermore, G. Faltings announced [F] and L. Ein proved [E2] that $H^i(\mathbb{P}^N, \mathcal{I}_X(1))$ for $1 \leq i \leq 3n - 2N$. However for *smooth* n-dimensional variety X in $\mathbb{P}^N$, Hartshorne's conjecture says that X is a complete intersection provided $N < \frac{3}{2}n$. Hence for smooth varieties in this dimension range *all* the cohomology groups $H^i(\mathbb{P}^N, \mathcal{I}_X(j))$ for $1 \leq i \leq n$ are expected to vanish. Thus, the above bounds for smooth codimension two varieties seem to be far from being sharp. Note that there are many non-projectively normal threefolds in $\mathbb{P}^5$ (of course, this case is not in the Hartshorne's conjecture range), see [Rao] and it is known that any smooth threefold X of degree d in $\mathbb{P}^5$ is k-normal for all $k \geq d - 4$, which is sharp as the Palatini scroll of degree 7 shows, for details see [K2].

The following conjecture was initiated by Castelnuovo in 1893 who solved the following problem for smooth curves in $\mathbb{P}^3$, see [Sz].

Regularity conjecture [EG], [GLP]. *Let X be a nondegenerate integral projective scheme of dimension n and degree d in $\mathbb{P}^N$ which is defined over algebraically closed field of characteristic zero.*

1. *X is m-normal for all $m \geq d - \operatorname{codim}(X)$.*
2. *X is m-regular for all $m \geq d - \operatorname{codim}(X) + 1$, i.e.,* $\operatorname{reg}(X) = \min\{m \in \mathbb{Z} : X \text{ is } m\text{-regular}\} \leq d - (N - n) + 1$.
3. *Classification of all extremal examples with geometric interpretations which make the bound best possible.*

It is natural to ask whether the degrees of all minimal generators of the saturated ideal of X are bounded by d. Note that X is set-theoretically an intersection of hypersurfaces (actually cones) of degree at most d.

More strongly, it has been conjectured that the degrees of all minimal generators are bounded by $d - e + 1$. An important role in the study of this question is played by the Castelnuovo-Mumford regularity $\operatorname{reg} X$. According to [EG], [Mu1], X is m-regular iff one of the following conditions holds:

1. $H^i(\mathbb{P}^N, \mathcal{I}_X(m - i)) = 0$ for all $i \geq 1$;
2. $H^i(\mathbb{P}^N, \mathcal{I}_X(j)) = 0$ for $i \geq 1$, $i + j \geq m$;
3. For all $k \geq 0$ the degrees of minimal generators of the k-th syzygy modules of the homogeneous saturated ideal I_X of X are bounded by $k + m$.

For a proof of these equivalences, see ([BM], Appendix). Attempts to bound the regularity of a projective variety $X \subset \mathbb{P}^N$ are motivated in part by the desire to bound the complexity of computing the syzygies and related invariants of X. There is no clear understanding of what to expect if X is singular(but reduced and irreducible).

The importance of m-regularity stems from the above equivalences and the following well-known results [Mu1]; If X is m-regular then X is cut out by hypersurfaces of degree m set-theoretically and scheme-theoretically. Furthermore, Hilbert polynomial and the Hilbert function of X have the same values for all $k \geq m - 1$.

This regularity conjecture including the classification of borderline examples was verified for integral curves ([C],[GLP]) and an optimal bound was also obtained for smooth surfaces ([P], [L]). For a historical remark and further results for higher dimensional smooth varieties, see [K1]. Roughly speaking, the varieties on the boundary of Regularity Conjecture are characterized by the property of having a $(d-(N-n)+1)$-secant line. (Clearly, $d-(N-n)+1$ is the largest possible number of intersections of a line with a nondegenerate variety of degree d by the generalized Bezout Theorem.) Note that if X has $(m+1)$-secant line, X can not be m-regular. In addition, when the locus of all m-secant lines of X is not so big for $m \le \dim(X)$ (e.g., X is defined by cubic polynomials or less and $\dim(X) \le 6$), we can get the Castelnuovo-Mumford regularity bounds less than or equal to $(d-(N-n)+1)$. So, the geometry of multisecant lines of a given smooth variety and vector bundle techniques (developed by Lazarsfeld, [L]) link the behavior of graded Betti numbers of the minimal graded free resolution of homogeneous coordinate ring of X to the vanishing of cohomology groups of certain vector bundles on smooth varieties. This is a connection between syzygies and geometry of a smooth variety. In particular, Kodaira-Kawamata-Viehweg vanishing theorem is useful in the case of smooth varieties, see [BEL], where we can see some important results bounding Castelnuovo regularity in terms of degrees of defining equations and their applications.

On the other hand, there are various approaches in the categories of monomial ideals, toric varieties, and locally Cohen-Macaulay varieties, see [PSt] [SV].

More open problems and Objectives

Based on the above discussion, we can list the following open problems and objectives which are related each other.

Peskine's conjecture. *Let $X^n \subset \mathbb{P}^{n+2}$ be a nondegenerate projective algebraic variety of dimension n over $\mathbb{C}$. No assumption is made on the singularities of X. We have $H^1(\mathbb{P}^{n+2}, \mathcal{I}_X(m)) = 0$ for $m \le n-2$ and $H^1(\mathbb{P}^{n+2}, \mathcal{I}_X(n-1)) = 0$ with a finite number of classifiable exceptions. Furthermore, one expects the exceptions to be singular in codimension $n-4$.*

Note that if $n=2$, the conjecture is well known, at least for smooth surfaces. The first statement is vacuous and the second statement is the well known assertion that all surfaces in $\mathbb{P}^4$ are linearly normal except the projection of the Veronese. For $n=3$, the statement is that all threefolds are linearly normal (Zak's theorem, at least for smooth varieties), and second that the only threefold in $\mathbb{P}^5$ that is not quadratically normal is the Palatini scroll of degree 7 which is still open. The Peskine's conjecture is extended by F. Zak to the following.

Zak's conjecture. *Let $X^n \subset \mathbb{P}^N$ be a nondegenerate projective algebraic variety (not necessary smooth) of dimension n, and let $\mathcal{I}_X$ be its sheaf of ideals.*

1. *$H^i(\mathbb{P}^N, \mathcal{I}_X(j)) = 0$ for $i \ge 1$, $j \ge 0$, $i+j < \frac{n}{N-n-1}$;*
2. *For $i \ge 1$, $j \ge 0$, $i+j = \frac{n}{N-n-1}$ it is possible to describe all varieties for which $H^i(\mathbb{P}^N, \mathcal{I}_X(j)) \ne 0$.*

F. Zak introduced a more geometric notion of *strong j-normality* and used the example of hypersurfaces to show how strong j-normality of the ambient projective space is inherited by the varieties of small codimension. Zak's approach is based on considering *multisecant lines* of the variety X and their behaviour under the Veronese maps. Using higher secant varieties of the Veronese embeddings of X, we introduce a natural filtration in $H^1(\mathbb{P}^N, \mathcal{I}_X(j))$ and show that this filtration degenerates in low codimensions. In particular, it seems like that Zak's conjecture helps us to attack Regularity conjecture in the case of singular varieties via cohomological methods which is still effective for smooth varieties even though we can not use Kodaira-Kawamata-Viehweg vanishing theorem.

Problems on multisecant lines. *Let $X \subset \mathbb{P}^N$ be a smooth nondegenerate n-dimensional projective variety, and let $S_m(X)$ be the locus of m-secant lines of X in $\mathbb{P}^N$. Assume that $n \leq 14$. Then one has $\dim S_{n+2-m} \leq n+1+m$, which gives us some information on "collinear" fibers of a generic linear projection of X to a hypersurface. For details, see J. Mather's theorem in* [K1] *and (dimension+2)-secant lemma in* [Ran3]. *These give us useful information on Regularity conjecture from the author's viewpoint.*

1. *Classify all surfaces in $\mathbb{P}^5$ whose trisecant lines is exactly four-dimensional. Note that almost all cases of smooth surfaces in $\mathbb{P}^5$ have three dimensional trisecant lines and it has been considered for a long time that there is no smooth surface in $\mathbb{P}^5$ having 4-dimensional trisecant lines but a few examples exist, see* [Dob].
2. *Classify all smooth threefolds in $\mathbb{P}^5$ whose all 4-secant lines do not fill up $\mathbb{P}^5$. We remark that all smooth surfaces in $\mathbb{P}^4$ whose trisecant lines do not fill up whole space was completely classified in* [Au] *and Z. Ran claimed that for a smooth threefold in $\mathbb{P}^N, N \geq 9$, the locus of 4-secant lines of X is at most 4-dimensional which implies Regularity conjecture for smooth threefolds,* [Ran2].

Objectives.

1. *To study the behavior of finite schemes under the Veronese maps;*
2. *To study geometrically the notion of j-normality;*
3. *To study multisecant lines of varieties of small codimension and the varieties swept out by them;*
4. *To study the varieties on the boundary of various conjectures, their resolutions and multisecant lines;*

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School of Mathematics, Korea Institute for Advanced Study, 207-43 Chungryangri-dong, Dongdaemoon-gu, Seoul 130-010, Korea

E-mail address: sjkwak@kias.re.kr