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DIFFERENTIABLE DYNAMICAL SYSTEMS

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Abstract. This is a survey article on the area of differentiable dynamical systems which are based on geometrical assumptions about the dynamical process; especially focused on the hyperbolic structure and the stability of diffeomorphisms which are key notions in differentiable dynamical systems.

(hyperbolicity) (stability)

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0. Introduction

,
(dynamical system).

(integral curve)
19 H. Poincare
. A.M. Lyapunov G.D. Birkhoff
(stability) , A.A. Andronov, J. Auslander, V.V. Nemysky, W. Gottschalk, G. Hedlund, V.V. Stepanov
.
1960
(differentiable dynamical system) S. Smale
, D.V. Anosov, L. Arnold, M. Peixoto, M. Hirsch, J. Palis, C. Pugh, R. Mane .
(topological dynamics) continuous action , M
Lie G(, $G = \mathbb{Z}$, or $\mathbb{R}$) differentiable action .
survey article
.
, regularity , , Jordan canonical form
, vector fields, C^r-, transversality ,
Banach Implicit function Contraction mapping
.

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Key words and phrases. , diffeomorphism, (flow), (hyperbolicity), Anosov system, Axiom A, (structural stability).

1 (, $G = \mathbb{Z}$) ,
2 (, $G = \mathbb{R}$) ,
3 .

1. Diffeomorphisms

M compact, G Lie, $Diff^r(M)$ C^r- M C^r-diffeomorphism (, $r \geq 1$). M G C^r-action C^r map $\phi : M \times G \to M$:

1. $\phi(p, 0) = p$, $p \in M$, 0 G ;
2. $\phi(\phi(p, s), t) = \phi(p, s + t)$, $p \in M$, $s, t \in G$.

$G = \mathbb{Z}$ C^r-action ϕ , M C^r-diffeomorphism $f \in Diff^r(M)$. $G = \mathbb{R}$ C^r-action ϕ M C^r-flow 2 .

$p \in M$ f (orbit) $\{f^n(p) : n \in \mathbb{Z}\}$, p (periodic) $n \in \mathbb{Z}^+$ $f^n(p) = p$. , $n = 1$ p (fixed point) . f $Per(f)$.

f (structure) .

M , vector fields (, $G = \mathbb{R}$) (flow) , flow cross-section cross-section diffeomorphism . diffomorphism dynamic , flow .

,
: , .

1. ?
2. ?
3. (C^r-perturbation) ?
4. dynamics (generic property)?
5. (pseudo-orbit) (real-orbit) ?

atractor , (expansivity), (hyperbolicity), (stability) , shadowing property .

-(nonwan- dering), (chain recurrent) , $p \in M$ - p U $m \in \mathbb{Z}$ $n(\geq m)$ $f^n(U) \cap U \neq \emptyset$, $p \in M$ $\epsilon > 0$ p p ϵ-pseudo-oribit . f - $\Omega(f)$, $CR(f)$.
$\overline{Per(f)} \subset \Omega(f) \subset CR(f)$.

$\Lambda(\subset M)$ $f \in Diff^r(M)$ (hyperbolic) , Λ M tangent bundle $T_\Lambda M$ subbundle E^s, E^u Whitney sum , f derivative map Tf E^s contraction E^u expansion . M

f Anosov diffeomorphism . Anosov diffeomorphsim
toral hyperbolic automorphism .
f dynamics nonwandering set $\Omega(f)$
$\Omega(f)$. $CR(f)$
hyperbolic $CR(f) = \overline{Per(f)}$, $\Omega(f)$ hyperbolic $\Omega(f) \neq \overline{Per(f)}$ (Dankner). 'Axiom A' , f Axiom A $\Omega(f)$ hyperbolic $\Omega(f) = \overline{Per(f)}$. 'generically' $\Omega(f) = \overline{Per(f)}$.

S. Smale f Axiom A , $\Omega(f)$ basic sets (compact, invariant, topologically transitive sets) (Spectral decomposition theorem). $CR(f)$.

hyperbolic set . Stable manifold theorem Shadowing theorem . Stable manifold theorem $\Lambda \subset M$ hyperbolic set Λ x stable manifold $\omega^s_\epsilon(x)$ unstable manifold $\omega^u_\epsilon(x)$, Shadowing theorem Λ pseudo-orbit real orbit shadow . Λ hyperbolic $f|_\Lambda$ expansive ([6], [11], [15], [16]).

f $Diff^r(M)$ (C^r)-perturbation , f f (C^r)-structurally stable , diffomorphism . system structurally stable . perturbation , C. Pugh Closing Lemma C^2 .

Anosov diffeomorphism structurally stable $Diff^r(M)$ open set C^0- dense. f hyperbolic fixed point p f derivative map T_pf local conjugate; , (Hartman-Grobman theorem). f Axiom A , x, $y \in \Omega(f)$ global stable manifold $\omega^s(x)$ global unstable manifold $\omega^u(y)$ transverse f strong transversality condition(STC) , J. Palis S. Smale conjecture .

f Axiom A STC f structually stable .

conjecture C. Robinson, J. Palis , 1987 R. Mane C^1 (, C^1-Stability Conjecture) . C^2 Closing Lemma .

diffeomorphism hyperbolicity stability . , attractor, Morse-Smale system, generic property, bifurcation theory ([1], [2], [6], [7], [11], [13], [15], [16]) .

2. Flows

M flow $\phi : M \times \mathbb{R} \to M$ M tangent vector field ; , $p \in M$

$$(*) \qquad X(p) = \frac{\partial \phi(p,t)}{\partial t}|_{t=0}$$

. $X(p)$ M p tangent vector, $\phi(p,t) :=$
$\phi_t(p)$ $(*)$ (solution). p flow ϕ
(orbit) $\{\phi_t(p) : t \in \mathbb{R}\}$ $(*)$.
M $(*)$ $X(p)$ C^r, $r \geq 1$, M compact
regularity M C^r-flow
.

flow(or vector field) diffeomorphism . n-
M diffeomorphism f $(n+1)$- $\widetilde{M}$
flow(or vector field) X_f Poincare map . flow X_f f
suspension flow , diffeomorphism f flow X_f
.

f structurally stable X_f structurally stable.
diffeomorphism flow orbit .
1 ; , nonwandering, chain recurrent, pseudo-orbit, hyperbolicity, expansiveness, shadowing property, structural stability, Axiom A, strong transversality condition, Stable manifold theorem, Stability Conjecture, Closing Lemma diffeomorphism dynamic
flow . ([2],
[6], [11], [16]).
, diffeomorphism dynamics flow dynamics , .
.

, f M Anosov diffeomorphism $\Omega(f) = M$
, ϕ M Anosov flow
$\Omega(\phi) \neq M$ J. Franks R. Williams . , C^1-Stability Conjecture diffeomorphism 1987 R. Mane
, flow (1996) S. Hayashi L. Wen
.

, diffeomorphism flow .

3. Our Interests in Differentiable Dynamical Systems

M $f \in Diff^r(M)$ hyperbolic f Anosov diffeomorphism
, Anosov diffeomorphism $\mathbb{R}^n$ linear map L eigenvalue 1 , L matrix , L
1 linear map L $T^n = \mathbb{R}^n/\mathbb{Z}^n$ toral automorphism.
M. Hirsch ([12], Problem 10-(a)).

Is the restriction of a diffeomorphism to a hyperbolic manifold Anosov?

J. Franks C. Robinson M 3
. , ([12], Problem 10-(b))
.

If f is Anosov and Λ is a C^1 compact f-invariant submanifold of M then is $f|_\Lambda$ Anosov?

M. Hirsch, C. Pugh, C. Robinson, R. Mane, J. Franks, A. Zeghib ([4], [8], [9], [19], [20]).
hyperbolic set shadowing property
.

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