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RANDOM GRAPHS AND QUASI-RANDOM GRAPHS: AN INTRODUCTION

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ABSTRACT. This article is written to spark novices' interest in the theory of quasi-random graphs. We introduce the notion of quasi-randomness, explain the gap between graphs and random graphs, and bridge this gap with the aid of quasi-random graphs.

1. INTRODUCTION

For each positive integer n, our sample space Ω_n consists of all labeled graphs G of order n with vertex set $\{1, \ldots, n\}$. Thus the cardinality of Ω_n is 2^N, where $N = \binom{n}{2}$. Also specified is a number p, with $0 < p < 1$, called the "edge probability". Then the probability function P for this sample space is defined by

$$Pr(G) = p^M (1-p)^{N-M},$$

where M is the number of edges of G. Thus the sample space consists of the outcome of N Bernoulli trials. If $p = 1/2$, our sample space Ω_n becomes a uniform space with each particular graph having probability $2^{-\binom{n}{2}}$. In this article, the edge probability p is always $1/2$.

A subset $\mathcal{A}_n$ of Ω_n describes a *graph property* Q if it is closed under isomorphism, i.e., if $G \in \mathcal{A}_n$ and $H \cong G$, then $H \in \mathcal{A}_n$. For example, if $\mathcal{A}_n$ consists of all connected graphs in Ω_n, then $\mathcal{A}_n$ describes the graph property of connectedness. It may happen that

$$Pr(\mathcal{A}_n) \to 1 \quad \text{as} \quad n \to \infty,$$

in which case we say *almost all graphs have property* Q or *the random graph has property* Q *almost surely*. And a typical graph in Ω_n, which we denote by $G_{1/2}(n)$, will have property Q with overwhelming probability as n gets large. For example, $G_{1/2}(n)$ is connected almost surely (see [Le00] or [Bo85]). For a much fuller discussion of these concepts, the reader can consult [Pa85], [Bo85] or [AIES92].

One would like to construct graphs that behave just like a random graph $G_{1/2}(n)$. Of course, it is logically impossible to construct a truly random graph. Thus Chung, Graham, and Wilson defined in [ChGW89] "quasi-random graphs", which simulate $G_{1/2}(n)$ without much deviation.

In this article we will try to present just enough about quasi-random graphs as they apply to graphs to illustrate bridging the gap between graphs and random graphs. To do this we introduce a graph property A_2, show this property is a

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"random graph property", i.e., almost all graphs have property A_2, and construct a graph having this property with the aid of quasi-random graphs.

Notice that almost all materials in this article refer to [ChGW89] and [Pa9x].

2. Notation and Definitions

First, we introduce some notation. Let $G = (V, E)$ denote a graph with vertex set V and edge set E. We use the notation $G(n)$ (and $G(n, e)$) to denote that G has n vertices (and e edges). For $X \subseteq V$, we let $X|_G$ denote the subgraph of G induced by X, and we let $e(X)$ denote the number of edges of $X|_G$. For $v \in V$, define

$$nd(v) = \{x \in V : \{v, x\} \in E\} \quad \text{and} \quad \deg(v) = |nd(v)|.$$

If $G' = (V', E')$ is another graph, we let $N^*_G(G')$ denote the number of labeled occurrences of G' as an induced subgraph of G. In other words, $N^*_G(G')$ is the number of injections $\lambda : V' \to V$ such that $\lambda(V')|_G \cong G'$. We will often just write $N^*(G')$ if G is understood. Finally, we let $N_G(G')$ denote the number of occurrences of G' as a (not necessarily induced) subgraph of G. Thus, if $G' = (V', E')$ then

$$N_G(G') = \sum_H N^*_G(H)$$

where the sum is taken over all $H = (V', E_H)$ such that $E_H \supseteq E'$.

Now, to define so-called quasi-random graphs, we list a set of graph properties which a graph $G = G(n)$ might satisfy.

$\mathbf{P_1(s)}$: For all graphs $M(s)$ on s vertices,

$$N^*_G(M(s)) = (1 + o(1))n^s 2^{-\binom{s}{2}}.$$

Let C_t denote the cycle with t edges.

$\mathbf{P_2(t)}$:

$$e(G) \geq (1 + o(1))\frac{n^2}{4}, \qquad N_G(C_t) \leq (1 + o(1))\Big(\frac{n}{2}\Big)^t.$$

Let $A = A(G) = (a(v, v'))_{v,v' \in V}$ denote the adjacency matrix of G, that is, $a(v, v') = 1$ if $\{v, v'\} \in E$, and 0 otherwise. Order the eigenvalues λ_i of A (which of course are real) so that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$.

$\mathbf{P_3}$:

$$e(G) \geq (1 + o(1))\frac{n^2}{4}, \qquad \lambda_1 = (1 + o(1))\frac{n}{2}, \qquad \lambda_2 = o(n).$$

$\mathbf{P_4}$: For each subset $S \subseteq V$,

$$e(S) = \frac{1}{4}|S|^2 + o(n^2).$$

$\mathbf{P_5}$: For each subset $S \subseteq V$ with $|S| = \lfloor \frac{n}{2} \rfloor$,

$$e(S) = \Big(\frac{1}{16} + o(1)\Big)n^2.$$

For $v, v' \in V$, define

$$s(v, v') = |\{y \in V : a(v, y) = a(v', y)\}|.$$

P_6:

$$\sum_{v,v'} \left| s(v,v') - \frac{n}{2} \right| = o(n^3).$$

P_7:

$$\sum_{v,v'} \left| |nd(v) \cap nd(v')| - \frac{n}{4} \right| = o(n^3).$$

Theorem 1. [ChGW89] *For $s \geq 4$ and even $t \geq 4$, all the properties are equivalent.* □

Definition 2. A family $\{G(n)\}$ of graphs (or simply a graph $G(n)$) is quasi-random if $G(n)$ satisfies any (and therefore, all) of these properties.

A weaker property of $G(n)$ is the following.

P_0:

$$\sum_{v} \left| \deg(v) - \frac{n}{2} \right| = o(n^2).$$

It follows easily that the following property is equivalent to P_0:

P_0': All but $o(n)$ vertices of G have degree $(1 + o(1))\frac{n}{2}$.
In this case we say that G is *almost regular*.

3. Background for the Definition

In this section we will step in the background of the definition. The philosophy of this definition can be described as follows:

1. Define various properties of graphs P_1, P_2, ... , P_k all shared by $G_{1/2}(n)$.
2. Show that all the properties defined are equivalent, i.e., if $G(n)$ has property P_i, then $G(n)$ has property P_j also.
3. Typically, it is easy to show that a particular family $\{G(n)\}$ satisfies some P_i.

We called these properties quasi-random.

Our spirit is to make quasi-random graphs imitate a typical random graph $G_{1/2}(n)$ without much deviation. Thus we would like to see how much quasi-random graphs deviate from $G_{1/2}(n)$. Recall that our sample space is Ω_n with edge probability $p = 1/2$.

$P_1(s)$: (Induced s-subgraph property) Let $M(s)$ be a graph on s vertices and s fixed. We know that almost all $G_{1/2}(n)$ contain $(1+o(1))n^s 2^{-\binom{s}{2}}$ induced copies of $M(s)$ as $n \to \infty$ [Bo85]. The content of $P_1(s)$ is that all of the $2^{\binom{s}{2}}$ labeled graphs $M(s)$ on s vertices occur asymptotically the same number of times in G.

$P_2(t)$: (t-cycle property) It is easy to see that for almost all random graphs $G_{1/2}(n)$,

$$e(G_{1/2}(n)) = (1 + o(1))\frac{\binom{n}{2}}{2} = (1 + o(1))\frac{n^2}{4}$$

and that almost all $G_{1/2}(n)$ contain

$$\frac{< n >_t}{2^t} = (1 + o(1))\left(\frac{n}{2}\right)^t$$

induced copies of C_t as $n \to \infty$, where $< n >_t$ denotes the falling factorial. Thus, the property $P_2(t)$ is a slight generalization of these two facts.

P_3: (Separated eigenvalue property) A result of Juhász [J78] shows that almost all random graphs $G_{1/2}(n)$ have

$$\lambda_1 = (1 + o(1))\frac{n}{2}$$

and

$$\lambda_2 = o(n^{\frac{1}{2}+\epsilon})$$

for any fixed $\epsilon > 0$. Thus, $\lambda_2 = o(n)$.

P_4: (Uniform edge density property) For almost all random graphs $G_{1/2}(n)$ and each subset $S \subseteq V(G_{1/2}(n))$,

$$e(S) = \frac{\binom{|S|}{2}}{2} = \frac{1}{4}|S|^2 + o(n^2).$$

P_5: (Uniform edge density property for bisectors) Once we take $|S| = \lfloor \frac{n}{2} \rfloor$ in the uniform edge density property, we have

$$e(S) = \Big(\frac{1}{16} + o(1)\Big)n^2$$

for each subset $S \subseteq V(G_{1/2}(n))$ with $|S| = \lfloor \frac{n}{2} \rfloor$.

P_6: (Sameness property) For almost all random graphs $G_{1/2}(n)$ and $v, v' \in V(G_{1/2}(n))$,

$$s(v, v') = (1 + o(1))\frac{n}{2} \quad \text{and} \quad |\{(v, v') : v, v' \in V(G_{1/2}(n))\}| = n^2.$$

Thus,

$$\sum_{v,v'} \Big|s(v, v') - \frac{n}{2}\Big| = n^2 o(n) = o(n^3).$$

P_7: (Common neighborhood property) For almost all random graphs $G_{1/2}(n)$ and $v, v' \in V(G_{1/2}(n))$,

$$|nd(v) \cap nd(v')| = (1 + o(1))\frac{n}{4} \quad \text{and} \quad |\{(v, v') : v, v' \in V(G_{1/2}(n))\}| = n^2.$$

Thus,

$$\sum_{v,v'} \Big||nd(v) \cap nd(v')| - \frac{n}{4}\Big| = n^2 o(n) = o(n^3).$$

P_0 and P_0': (Degree property) For almost all random graphs $G_{1/2}(n)$ and $v \in V(G_{1/2}(n))$,

$$\deg(v) = (1 + o(1))\frac{n}{2}.$$

4. Example

Our next illustration revolves around a purely graph-theoretic question: does there exist a graph G with the property that for any disjoint pair of 2-subsets of vertices, say $\{u_1, u_2\}$ and $\{v_1, v_2\}$, there exists a vertex w adjacent to both u_1 and u_2 but not adjacent to v_1 and v_2? Let us call this graph property A_2. The reader is sure to see that the construction of such a graph may be a formidable problem. We showed in [Le00] the following theorem. However, we sketch again the proof for the sake of further explanation.

Theorem 3. *There exist graphs of every order $n \geq 345$ with property A_2. Furthermore, almost all graphs have property A_2.*

Proof. Let $\mathcal{A}_n$ be the set of graphs in our sample space Ω_n with property A_2. Our goal is to show that $Pr(\mathcal{A}_n) > 0$ for some value of n. We define the random variable $X(G)$ to be the number of disjoint pairs of 2-subsets for which there is no other vertex adjacent to both vertices of the first 2-subset and to neither vertex of the second 2-subset. Then the event $\mathcal{A}_n$ consists of all graphs G in Ω_n with $X(G) = 0$, that is,

$$Pr(\mathcal{A}_n) = Pr(X = 0).$$

Since X has non-negative values, we have

$$Pr(X \geq 1) \leq E[X].$$

And since X has only integer values, $E[X] < 1$ implies $Pr(X = 0) > 0$. Here is the formula for the expectation:

$$E[X] = \binom{n}{2, 2, n-4} \left(1 - p^2(1-p)^2\right)^{n-4}. \tag{4.1}$$

A little calculus shows that the right side (4.1) is smallest for $p = 1/2$ and so we would like a solution of the inequality:

$$\frac{n(n-1)(n-2)(n-3)}{4} \left(1 - \frac{1}{16}\right)^{n-4} < 1. \tag{4.2}$$

After a few minutes of computer job we found that for $n = 344$, the left side of (4.2) was $1.0157\cdots$ but at $n = 345$, it drops to $.9634\cdots$. This shows that for every $n \geq 345$, there exists a graph with property A_2. Furthermore, the left side of (4.2) regarded as a function of n approaches 0 as $n \to \infty$. Hence $Pr(X = 0) \to 1$ as $n \to \infty$ and so the overwhelming majority of graphs have property A_2 when n is large. □

But the method provides no help in constructing examples except by random creation. It follows from (4.2) that with $n = 400$,

$$Pr(X \geq 1) < .050151\cdots$$

and so $Pr(X = 0)$ is at least .95. Otherwise put, at least 95% of all graphs have property A_2. Suppose we create a random graph H with $n = 400$ and edge probability $p = 1/2$. The worst case complexity of an algorithm that tests H for property A_2 is $\mathcal{O}(n^5)$. So an example could be found in reasonable time. Thus we are faced with the irony that examples are omnipresent but the method provides no way to describe in a constructive way just one example for each value of n, where n is large. We summarize the issue raised as follows:

Question: Construct a graph with property A_2 of each order for which they exist.

To answer this question, try to dig out quasi-random graphs and select one with property A_2. Fortunately a family, called the Paley graphs, turns out to be a quasi-random graph having property A_2.

To define the Paley graphs, let q be a prime power congruent to 1 modulo 4. Hence -1 is a square (quadratic residue) in the finite field $\mathbb{F}_q$. The Paley graph P_q has the field $\mathbb{F}_q$ as its vertex set and vertices x and y are adjacent whenever $x-y$ is a square. For details, see [BoT81] and any elementary number theory text.

Theorem 4. [ChGW89] *The family of the Paley graphs P_q is quasi-random.*

Proof. First, observe that a vertex z is adjacent to both, or non-adjacent to both, of a pair x, y of distinct vertices of P_q if and only if the quotient $(z-x)/(z-y)$ is a square. But for any of the $(q-1)/2-1$ squares a other than 1, there is unique z such that

$$\frac{z-x}{z-y} = 1 + \frac{y-x}{z-y} = a.$$

Thus, $s(x,y) = (q-3)/2$ so that P_6 holds. □

Theorem 5. [BoT81] *For all q sufficiently large, the Paley graphs P_q have property A_2.* □

Theorem 5 means that there is a positive integer N such that P_q has property A_2 for all $q \geq N$ and q is a prime power congruent to 1 modulo 4. But a question still remains: for what value of q, does the Paley graph P_q have property A_2? Ron Read [Re93] wrote a computer program to test Paley graphs for property A_2 and found that P_{61} was the smallest. Let a_2 be the smallest integer for which there is a graph with a_2 vertices and property A_2. Thus Read showed that $a_2 \leq 61$. How about a decent lower bound? There is also the problem of determining all the values of n between a_2 and 344 for which graphs exist with property A_2.

We should not conclude this section without a word of caution: there are perfectly nice quasi-random families that are a bit too quasi to have property A_2. For each n, let V_n consist of all the n-subsets of $\{1, \ldots, 2n\}$. Then G_n is the graph with vertex set V_n in which x and y in V_n are adjacent iff $|x \cap y|$ is even. It can be shown that this family satisfies the quasi-random criterion [ChGW89]. However, none of the G_n have property A_2. To see this let $x = \{1, \ldots, n\}$ and $y = \{n+1, \ldots, 2n\}$. For any vertex $w \in V_n$, we have

$$n = |w| = |w \cap x| + |w \cap y|. \tag{4.3}$$

If G_n has property A_2, there must be a vertex w_1, adjacent to both x and y. So $|w_1 \cap x|$ and $|w_1 \cap y|$ are both even. Equation (4.3) implies that n is even. But there must also be a vertex w_2 adjacent to x but not y. Thus $|w_2 \cap x|$ is even and $|w_2 \cap y|$ is odd. Now equation (4.3) implies n is odd, a contradiction.

5. Suggestions for Readers

For further study of quasi-random graphs, we introduce some materials. Elementary graph theory texts might be [Ha69], [ChL86], and [We96]. For an easy introduction to random graphs with an appendix of basic probability theory, the reader may see the book [Pa85]. And for a more advanced treatment there are the research monographs [Bo85], [AlES92], and [Ko99]. It is also essential to become familiar with a standard probability text such as [Fe57]. For a full discussion of quasi-random graphs, the reader can consult [ChGW88], [ChGW89], and [ChG92a] first. And next discuss with [Ch90], [Ch91], [ChG90], [ChG91a], [ChG91b], [ChG91c], and [ChG92b].

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