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REMARKS ON CODIMENSION TWO SUBVARIETIES OF $\mathbb{P}^n$

YOUNGOOK CHOI

Abstract. Hartshorne conjectured that any smooth projective subvariety in $\mathbb{P}^N$ is a complete intersection if its dimension is bigger than two times of its codimension. Since complete intersections are the simplest projective subvarieties, detecting them among all subvarieties is an important question in algebraic geometry. In this note, we give a brief survey of recent progress for this conjecture in codimension two case and discuss on related problems.

1. Introduction

Let X be an n-dimension smooth projective subvariety of $\mathbb{P}^N$ and $I(X)$ its homogeneous ideal of X. X is called a *complete intersection* if the number of minimal generators of $I(X)$ is equal to the codimension of X. Since complete intersections are the simplest subvarieties among all of them, detecting complete intersections is an important questions in algebraic geometry and far from complete answers until now except some trivial cases such as hypersurfaces in projective space. First achievement related to this problem is so-called "Barth-type" theorem [Ba1]. It says that a smooth projective variety whose codimension is small compared to its dimension looks like homologically a complete intersection. It is a generalization of the classical theorem of Lefschetz and with no known examples other than complete intersections it led Hartshorne to conjecture the following theorem in [Ha1].

Conjecuture 1.1 (Hartshorne)**.** *Any smooth projective subvariety X is in fact a complete intersection if* $\dim(X) > 2 \cdot \operatorname{codim}(X)$.

X is called *subcanonical* if the canonical sheaf of X is some twist of structure sheaf, i.e. $w_X = \mathcal{O}_X(k)$ for some $k \in \mathbb{Z}$. Due to the Serre's construction of rank two vector bundles corresponding to codimension two subcanonical subvarieties, Hartshorne's conjecture (in codimension two case) is equivalent to the following statement : there are no rank two *indecomposable* vector bundles on $\mathbb{P}^N$ for $N \geq 6$. We remark that all of the smooth varieties of codimension two in P^N for $N \geq 6$ are subcanonical [BL]. So far Horrocks-Mumford bundle [HM] on $\mathbb{P}^4$ is the only known rank two indecomposable vector bundle on $\mathbb{P}^N$ for $N \geq 4$ and the existance of such a vector bundle on $\mathbb{P}^5$ is a big question. Problem lists in [Sch1] give challengeable questions about the vector bundles and low codimensional submanifolds on projective space.

There have been mainly two approaches to solve Hartshorne's conjecture in the last twenty-five years. One [BC, Ba2, EF, HS, Ra] is to bound the degree of X or a

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hypersurface containing X and the other [Ar, CK, Fa, Ne] is to bound the number of minimal homogeneous polynomials which cut out X scheme theoretically. An exact definition of scheme-theoretic intersection will be given in section 3. The ideal $I_{sch}(X)$ generated by these homogeneous polynomials is called a *schematic ideal* of X. $I_{sch}(X)$ may not be equal to saturated homogenous ideal $I(X)$, but they are coincide in high degree. Any scheme X in $\mathbb{P}^N$ which is locally a complete intersection is a scheme-theoretic intersection of at most $N+1$ hypersurfaces [Fu]. So one can bound the number of minimal generators of $I_{sch}(X)$ by $N+1$ for any scheme X in $\mathbb{P}^N$. Notice that there is no upper-bound of the number of minimal generators of $I(X)$.

2. Bounding the degree of X

The first estimation on the degree of X to be a complete intersection was given by Barth and Van de Ven [BV]. They proved that X is a complete intersection if $\deg(X) \le \frac{1}{4}(N+5)$. Schneider [Sch2] established a quadratic estimation for subvarieties with semi-stable normal bundle and Ran [Ra] improved Barth-Van de Ven's bound for $\deg(X) \le N-2$. His main result is the following lemma.

Lemma 2.1 (Ran)**.** *Let $X \subset \mathbb{P}^N$ be a codimension two subcanonical subvariety. Set $e(k) = k^2 - c_1 k + c_2$, where c_i are the Chern classes of the rank two associated vector bundle. Fix $k \le N-2$ and assume $e(i) \ne 0$ for all $0 \le i \le k$, then there exists a $(k+1)$-secant line to X through a general point of $\mathbb{P}^N$. In particular $h^0(\mathcal{I}_X(k)) = 0$.*

This lemma shows that if X is contained in a hypersurface of degree less than or equal to $N-2$, then X is a complete intersection. To improve Barth-Van de Ven's bound, Ran applied vector bundle technique to the above lemma. The technique he used is one of the powerful methods in solving Hartshorne's conjecture so far. Subsequently, Holme and Schneider [HS] and Ballico [BC] improved Ran's bound. We can summarize the known results from this approach as a following theorem.

Theorem 2.2. *Let $X \subset \mathbb{P}^N$ be a smooth codimension two subvariety of degree d and s the minimal degree of an hypersurface containing X. Also, assume that X is subcanonical, i.e. $w_X = \mathcal{O}_X(e)$.*

If $e \le N+1$ or $d < (N-1)(N+5)$ or $s \le N-2$, then X is a complete intersection.

On the other hand, the singular locus of the hypersurface $S \subset \mathbb{P}^N$ containing X gives another information for complete intersections. By the Lefschetz's theorem, if X is a Cartier divisor of S for $N \ge 4$, then X is a complete intersection of S with another hypersurface. Therefore if there exists any single smooth hypersurface containing X, X is a complete intersection. Ellia and Franco [EF] did an analysis of the singular locus of S and gave the following theorem.

Theorem 2.3. *Let $X \subset \mathbb{P}^6$ be a smooth codimension two subvariety. If $s \le 5$ or $d \le 73$, then X is a complete intersection.*

3. Bounding the number of generators of I_{sch}

An n-dimensional projective variety $X \subset \mathbb{P}^N$ is said to be the scheme-theoretic intersection of k equations $f_1, f_2, \cdots, f_k$ if $X = H_1 \cap \cdots \cap H_k$ *as schemes* where H_i

is a hypersurface defined by an equation f_i. It is equivalent to say that there is a surjection

$$\bigoplus_{i=1}^{k} \mathcal{O}_{\mathbb{P}^n}(-d_i) \xrightarrow{\phi} \mathcal{I}_X \to 0 \tag{3.1}$$

with $d_i = \deg f_i$, where $\mathcal{I}_X \subset \mathcal{O}_{\mathbb{P}^n}$ is the sheaf of ideals defining X. As mentioned in the introduction, X can be cut out at most $N+1$ equations.

If $k = 2$, it is easy to prove that X is a complete intersection [Ha, p188]. X is called a *quasi-complete intersection* if $k = 3$. The first criterion for k, n, N to be sure that X is a complete intersection was given by Faltings [Fa]. He showed that X is a complete intersection if $k \leq N/2$ or $\max(k, \frac{3}{4}N) \leq n$. Therefore every quasi-complete intersection of codimension two in $\mathbb{P}^N$ for $N \geq 6$ is a complete intersection. Recently, there are two improvements of Faltings result.

Theorem 3.1 ([Ne]). *Let X be a smooth subvariety in $\mathbb{P}^N$. Assume that $\dim(X) \geq \frac{3}{4}N - \frac{1}{2}$. If X can be defined by k equations and $k \leq \dim(X) + 1$, then X is a complete intersection.*

Theorem 3.2 ([Ar]). *Let X be a subcanonical locally complete intersection subvariety of codimension two in $\mathbb{P}^N$ ($N \geq 3$). If X can be defined by k equations and $k \leq N - 1$, then X is a complete intersection.*

However, both theorems do not hold for every smooth subvarieties of codimension two in $\mathbb{P}^4$ or $\mathbb{P}^5$. With respect to Netsvetaev's Theorem, there are infinitely many threefolds in $\mathbb{P}^5$ which can be cut out scheme-theoretically by *four* equations, but not a complete intersection [CK]. By Arsie's theorem all these examples should not be subcanonical.

For Arsie's theorem, we might ask whether a subcanonical variety of codimension two is a complete intersection or not when it is defined by N number of equations. The answer is *no* because a smooth abelian surface of degree 10 in $\mathbb{P}^4$ is cut out by four equations [CK]. Such a smooth abelian surface is a zero set of a section of the indecomposable rank 2 Horrocks-Mumford bundle, so it is clearly subcanonical but not a complete intersection by numerical reason of chern classes of Horrocks-Mumford bundle. Thus Arsie's theorem can not be extended to a smooth subcanonical surface in $\mathbb{P}^4$ defined by four equations. It gives a nice question for threefolds in $\mathbb{P}^5$.

question 3.3. *Is a subcanonical smooth threefold in $\mathbb{P}^5$ defined by five equations a complete intersection?*

4. Related problems

$X \subset \mathbb{P}^N$ is called *linearly normal* if X is not the projection of a variety in $\mathbb{P}^{N+1}$, not lying in any $\mathbb{P}^N$. In terms of sheaf cohomology, this says that $H^1(P^N, \mathcal{I}_X(1)) = 0$. Likewise, X is said *k-normal* if $H^1(P^N, \mathcal{I}_X(k)) = 0$. X is *projectively normal* if X is k-normal for every k. Since, for a complete intersection X, all the cohomology groups $H^i(P^N, \mathcal{I}_X(j))$ for $1 \leq i \leq N$ and $j \in \mathbb{Z}$ vanish, X is projectively normal. Especially X is linearly normal, so X can not be realized as a projection of a variety sitting in a higher dimensional projective space. Ein [Ei] proved the converse, i.e. if X is projectively normal and subcanonical of codimension two, then X is a complete intersection.

Hartshorne conjectured in [Ha1] that n-dimensional nonsingular variety $X \subset \mathbb{P}^N$ is a linearly normal if $n > \frac{2}{3}(N-1)$. The solution of this conjecture was obtained by

Zak [Za] who gives a remarkable method of using multi-secant lines. In codimension two case, there is an important theorem of Severi [Se] for the linear normality which implies that any codimension two smooth subvariety $X \subset \mathbb{P}^N$ for $N \geq 4$ is linearly normal except a projection of Veronese surface.

Theorem 4.1. *Let Y be a surface in $\mathbb{P}^5$, not contained in any hyperplane, whose generic projection into $\mathbb{P}^4$ is nonsingular. Then, up to an automorphism of P^5, Y is the Veronese surface, which is a quadratic embedding of $\mathbb{P}^2$ into $\mathbb{P}^5$.*

The Veronese surface in $\mathbb{P}^4$ can be described as the dependency locus of a natural map

$$\mathcal{O}_{P^4}^{\oplus 3} \longrightarrow \Omega^1_{P^4}(2).$$

Similarly, we can think of a threefold in $\mathbb{P}^5$ as dependency locus of a map

$$\mathcal{O}_{P^5}^{\oplus 4} \longrightarrow \Omega^1_{P^5}(2).$$

This threefold X is called *Palatini scroll* and it is the only known example of not being quadratically normal. The conjeuture of Peskine and Van de Van is that all smooth threefolds in $\mathbb{P}^5$ are quadratically normal except the Palatini scroll. This is a part of a general following conjecture.

Conjecuture 4.2 (Peskine and Zak)**.** *Let X be a nondegenerate, not necessary smooth, projective variety of codimension e in $\mathbb{P}^N$.*

1. $H^i(\mathbb{P}^N, \mathcal{I}_X(j)) = 0$ *for* $i \geq 1$, $j \geq 0$, $i + j < \frac{\dim(X)}{e-1}$;
2. *For* $i \geq 1$, $j \geq 0$, $i + j = \frac{\dim(X)}{e-1}$, *it is possible to describe all varieties for which* $H^i(\mathbb{P}^N, \mathcal{I}_X(j)) \neq 0$

In [CK], they proved the vanishing of the cohomology groups of small twisting of ideal sheaf for a smooth, not necessarily subcanonical threefold X defined by at most four equations. This contributes to the Peskine-Van de Ven conjecture. The exact statement is the following.

Theorem 4.3. *Let X be a smooth threefold in $\mathbb{P}^5$ with an ideal sheaf $\mathcal{I}_X$. Suppose that X is cut out scheme theoretically by at most four equations.*

(a) $H^1(\mathcal{I}_X(k)) = 0$ *for* $k \leq 3$
(b) *If the following two multiplicative maps*

$$H^0(\mathcal{O}_{\mathbb{P}^5}(1)) \otimes H^1(\mathcal{O}_X(1)) \to H^1(\mathcal{O}_X(2))$$

$$H^0(\mathcal{O}_{\mathbb{P}^5}(1)) \otimes H^2(\mathcal{O}_X) \to H^2(\mathcal{O}_X(1))$$

are injective, then $H^1(\mathcal{I}_X(4)) = 0$

The method they used is to check how the small number of defining equations of X affects Castelnuovo's regularity of the vector bundle $\mathcal{E}_k$ which is the kernel of the map ϕ in (3.1) and to apply the Kodaira-Le Potier vanishing theorem for ample vector bundles.

Finally, you can consult a good survey paper [Ka] for more related notions such as projective normality, regularity, etc.

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School of Mathematics Korea Institute for Advanced Study 207-43 Cheongryangri-Dong Dongdaemun-Gu Seoul Korea

E-mail address: ychoi@math.kias.re.kr