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A SURVEY ON EQUIVARIANT K-THEORY OF REPRESENTATION SPHERES

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ABSTRACT. We review recent results on equivariant K-theory of representation spheres which play as the coefficient ring of equivariant cohomology theory indexed by all virtual real representations instead of integers.

1. Equivariant K-theory

Equivariant K-theory is one of the main topics in equivariant algebraic topology which concerns the study of algebraic invariants of spaces with group actions. It has been widely applied to various branches of mathematics, even to string theory in physics (see [AS89] for instance). We refer the reader to [Ati67, Seg68, Kar78] for a general reference of (equivariant) K-theory.

Throughout, G is a compact Lie group although G can be any topological group in several cases. Given a compact G-space X, the *equivariant K-theory* of X, denoted by $K_G(X)$, is defined to be the Grothendiek group of finite dimensional G-vector bundles over X. Then $K_G(X)$ has a natural ring structure induced by tensor product of G-vector bundles.

If X has a G-fixed basepoint $*$, the *reduced K-theory* of X, denoted by $\widetilde{K}_G(X)$, is defined to be the kernel of the restriction homomorphism $K_G(X) \to K_G(*)$ which is induced by the inclusion map $* \hookrightarrow X$. Since any complex representation of G defines a trivial bundle over X, the natural morphism from the complex representation ring $R(G) = K_G(*) \to K_G(X)$ makes $K_G(X)$ and $\widetilde{K}_G(X)$ into $R(G)$-algebras (although there is no identity element in $\widetilde{K}_G(X)$).

For a locally compact G-space X, we define $K^0_G(X) \equiv \widetilde{K}_G(X^+)$ where X^+ denotes the one-point compactification of X, and $K^1_G(X) \equiv \widetilde{K}_G(S^1 \wedge X^+)$ where $S^1 \wedge X^+$ is the reduced suspension of X^+. If X is compact, X^+ is the union of X and a disjoint G-fixed basepoint, so that $K^0_G(X)$ coincides with $K_G(X)$ defined above.

2. Representation spheres

Let V be a real representation of G. We have the unit sphere $S(V)$ and the one-point compactification S^V, G acting trivially on the point at infinity which is taken as the basepoint of S^V. Any integer $n \geq 0$ can be viewed as the trivial representation $\mathbb{R}^n$ of G, so that the standard sphere S^n is a special case of our

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definition. Note that S^V is G-homeomorphic to $S(1 \oplus V)$ and that $S^{1\oplus V}$ is G-homeomorphic to $S^1 \wedge S^V$.

A standard sphere S^n with a linear G-action is usually called a *representation sphere*. J. P. May pointed out in the book [May96, Chapter IX] the importance of representation spheres as follows. "The full richness of equivariant homotopy and (co)homology theory comes from the interplay of homotopy theory and representation theory that arises from the consideration of spheres with non-trivial actions by G. In principle, it might seem reasonable to allow arbitrary G-actions. However, a closer inspection of the role of spheres in nonequivariant topology, both in manifold theory and in homotopy theory, gives the intuition that we should restrict to the linear spheres that arise from representations."

3. Main objective

As one of (stable) equivariant cohomology theories, $RO(G)$-graded cohomology theory $h^*_G(X)$ has been considered a full equivariant generalization of ordinary singular cohomology, which is built in the interplay relating the Burnside ring, the real character ring $RO(G)$, and G-homotopy theory (see [LMM81] or [May96] for more details). The main difference from the other equivariant cohomology theories is that the coefficient ring is indexed by all virtual real representations instead of integers.

Equivariant K-theory is known as one of the adequate examples in $RO(G)$-graded cohomology theory, simply set $h^V_G(X) \equiv \widetilde{K}_G(S^V \wedge X)$ for real representations V of G, and extend the grading to all of $RO(G)$ by setting $h^{V-W}_G(X) = h^V_G(S^W \wedge X)$. Here, $h^*_G(S^0) = \widetilde{K}_G(S^*)$ is the coefficient ring of this $RO(G)$-graded cohomology.

The aim of this article is to analyze the $R(G)$-module (and even the $R(G)$-algebra) structure of $K^0_G(V) \equiv \widetilde{K}_G(S^V)$ and $K^1_G(V) \equiv \widetilde{K}_G(S^{1\oplus V})$ for each real representation V of G.

4. The case of complex representations

Equivariant Bott periodicity is the most important theorem in equivariant K-theory and is even more extraordinary than its nonequivariant counterpart.

Theorem (Thom isomorphism)**.** *Let E be a complex G-vector bundle over a locally compact G-space X. Then there is a natural Thom isomorphism*

$$\varphi\colon K^0_G(X) \cong K^0_G(E).$$

Let $\lambda(V)$ denote the alternating sum of exterior powers

$$\lambda(V) = 1 - V + \textstyle\bigwedge^2 V - \cdots + (-1)^{\dim V} \textstyle\bigwedge^{\dim V} V \in R(G),$$

and let $e_V\colon S^0 \to S^V$ be the based map sending the non-basepoint to 0. Taking $X = *$ and $E = V$, we have $\varphi\colon R(G) = \widetilde{K}_G(S^0) = K^0_G(*) \cong K^0_G(V) = \widetilde{K}_G(S^V)$. Conversely, e_V induces

$$e^*_V\colon \widetilde{K}_G(S^V) \to \widetilde{K}_G(S^0) = R(G).$$

Theorem (Bott periodicity)**.** *Let V be a complex representation of G and X a compact G-space. Then we have an isomorphism*

$$\varphi\colon \widetilde{K}_G(X^+) = K^0_G(X) \cong K^0_G(V \times X) = \widetilde{K}_G(S^V \wedge X^+).$$

Moreover, $e^*_V(b_V) = \lambda(V)$ where $b_V = \varphi(1) \in \widetilde{K}_G(S^V)$.

Since $V = \mathbb{R}^2 = \mathbb{C}$, equivariant K-theory is periodic with period 2 so that we may take

$$K_G^{2n}(X) \equiv K_G^0(X) \qquad \text{and} \qquad K_G^{2n+1}(X) \equiv K_G^1(X)$$

for all integers n. Note in particular that the coefficient ring is $K_G^0(*) = \widetilde{K}_G(S^0) = R(G)$ in even degrees and $K_G^1(*) = \widetilde{K}_G(S^1) = R(G) \otimes \widetilde{K}(S^1) = 0$ in odd degrees.

We have known the $R(G)$-algebra structure of $K_G^0(V)$ and $K_G^1(V)$ when V is a complex representation of G, that is,

$$K_G^0(V) = \widetilde{K}_G(S^V) \cong R(G) \quad \text{and} \quad K_G^1(V) = \widetilde{K}_G(S^{1\oplus V}) \cong 0.$$

Moreover, if $V = V' \oplus W$ for some complex representation W of G, then

$$K_G^0(V) = \widetilde{K}_G(S^V) = \widetilde{K}_G(S^{V'\oplus W}) = \widetilde{K}_G(S^{V'} \wedge S^W) \cong \widetilde{K}_G(S^{V'}) = K_G^0(V')$$

by the Bott periodicity theorem, and similarly $K_G^1(V) \cong K_G^1(V')$. Therefore it suffices to consider real representations V of G such that each irreducible component of V is not a complex representation, i.e., absolutely irreducible, and they are mutually non-isomorphic, since $2V$ can be viewed as a complex representation.

5. The case of one-dimensional real representations

Let V be a real representation of G of dimension one. If V is trivial, i.e., $V = \mathbb{R}^1$, then we already know that

$$K_G^0(\mathbb{R}^1) = \widetilde{K}_G(S^1) = 0 \quad \text{and} \quad K_G^1(\mathbb{R}^1) = \widetilde{K}_G(S^2) \cong R(G).$$

On the other hand, the case that V is nontrivial was considered by Yang in [Yan95, Theorem A and B] for finite cyclic groups G. His proof was quite long but elementary, and the calculation of $K_G^1(V)$ turned out to be false while the author and collaborators were generalizing his results for compact Lie groups G.

The $R(G)$-module structure of $K_G^0(V)$ was drawn from the work [CKMS00, Theorem 10.1] on classification of equivariant complex vector bundles over a circle by the author and collaborators. Applying the same technique Masuda and the author calculated in [CM00, Theorem 1.1] the $R(G)$-algebra structure of $K_G^1(V)$. In the following we state the results.

Let $\rho_V\colon G \to O(1) = \{\pm 1\}$ be the surjective homomorphism corresponding to the nontrivial representation V. Since the kernel of ρ_V, denote by H, is normal in G, there is a natural G-action on $R(H)$ given by conjugation. More precisely, let χ be a complex representation of H. Setting ${}^g\chi(h) = \chi(g^{-1}hg)$ for $h \in H$ we have a complex representation ${}^g\chi$ of H for each $g \in G$. In fact, the G-action on $R(H)$ factors through a G/H-action on $R(H)$ since ${}^h\chi = \chi$ for all $h \in H$.

Choose and fix an element $b \in G \setminus H$. Since G/H is of order two, ${}^b\chi = {}^g\chi$ for all $g \in G \setminus H$ so that ${}^b\chi$ is independent of the choice of the element $b \in G \setminus H$. Note that $R(H)$ has a canonical $R(G)$-module structure given by $\varphi \cdot \chi = \operatorname{res}_H \varphi \otimes \chi$ for $\varphi \in R(G)$ and $\chi \in R(H)$.

Theorem. *Let V be a nontrivial one-dimensional real representation of G and let $\rho_V\colon G \to O(1) = \{\pm 1\}$ be the corresponding surjective homomorphism. Then*

(1) $K_G^0(V) = \widetilde{K}_G(S^V)$ *is isomorphic to the ideal in $R(G)$ generated by $(1-\rho_V)\otimes \mathbb{C}$.*

(2) $K^1_G(V) = \widetilde{K}_G(S^{1\oplus V})$ *is isomorphic to the* $R(G)$*-module consisting of the elements* $\chi - {}^b\chi$ *in* $R(H)$ *for all complex representations* χ *of* H*. Moreover, the ring structure on* $K^1_G(V)$ *is given by* $xy = 0$ *for* $x, y \in K^1_G(V)$*.*

6. $K^*_G(V)$ is torsion free

¿From now on we are mainly interested in the group structure of $K^*_G(V)$ which is obviously abelian. We have two results on torsion freeness of $K^*_G(V)$. The first one was proved by Yang in [Yan95, Theorem F] that $K^*_G(V)$ can only have 2-torsion for compact Lie groups G.

The following result is the second one shown by Karoubi in [Kar00, Theorem 1.1]. Having general knowledge on equivariant K-theory of Banach category [Kar78, Chapter II] may be helpful to read his proof.

Theorem. *Let* G *be a finite group and* V *a real representation of* G*. Then the group* $K^*_G(V)$ *is free abelian of finite rank.*

It is a famous result of McClure [McC86, Theorem A] that equivariant K-theory is detected on finite elementary subgroups. A finite group is called *elementary* if it is a direct product of a p-group and a cyclic group having order prime to p.

Theorem. *Let* G *be a compact Lie group and* X *a finite* G*-CW complex. If* $x \in K_G(X)$ *restricts to zero in* $K_H(X)$ *for all finite elementary subgroups* H *of* G *then* $x = 0$*.*

Combining these two results we deduce that $K^*_G(V)$ is torsion free for compact Lie groups G.

7. Delocalized equivariant cohomology

Let G be a finite group and X a locally compact, Hausdorff, and paracompact G-space. For each $g \in G$, let $[g]$ denote the conjugacy class of g in G and let $C(g)$ denote the centralizer of g in G. Note that $C(g)$ acts on X^g, the set of g-fixed points in X, and that there is a homeomorphism $X^g/C(g) \xrightarrow{\cong} X^{g'}/C(g')$ between the quotient spaces if $g' \in [g]$.

According to delocalized equivariant cohomology [BC88, Theorem 1.19] there is a generalized Chern character isomorphism of $\mathbb{C}$-vector spaces,

$$K^0_G(X) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^0(X, G) \equiv \prod_{n \in \mathbb{Z}} \Big[\bigoplus_{[g]} \check{H}^{2n}_c(X^g/C(g); \mathbb{C}) \Big],$$
$$K^1_G(X) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^1(X, G) \equiv \prod_{n \in \mathbb{Z}} \Big[\bigoplus_{[g]} \check{H}^{2n+1}_c(X^g/C(g); \mathbb{C}) \Big],$$

where $[g]$ runs through all conjugacy classes in G. Here $\check{H}^*_c(X^g/C(g); \mathbb{C})$ denotes the Čech cohomology of $X^g/C(g)$ with compact supports.

In particular, if V is a real representation of G, we have

$$\check{H}^*_c(V^g/C(g); \mathbb{C}) \cong [\check{H}^*_c(V^g; \mathbb{C})]^{C(g)} \cong [\widetilde{H}^*(S^{V^g}; \mathbb{C})]^{C(g)}$$

where $[\ \ ]^{C(g)}$ denotes the $C(g)$-invariant part. For each $h \in C(g)$, the h-action on V^g induces a map $f_h \colon S^{V^g} \to S^{V^g}$. Note that all the cohomology classes in $\widetilde{H}^{\dim V^g}(S^{V^g}; \mathbb{C}) \cong \mathbb{C}$ are $C(g)$-invariant if the degree of f_h is equal to 1, i.e., the determinant of the h-action on V^g is positive, for all $h \in C(g)$. Otherwise $[\widetilde{H}^{\dim V^g}(S^{V^g}; \mathbb{C})]^{C(g)} = 0$.

We say that a conjugacy class $[g]$ in G is (positively) *oriented* if the determinant of the h-action on V^g is positive for all $h \in C(G)$. A conjugacy class $[g]$ is called *even* or *odd* according as the dimension of V^g is even or odd, respectively. Finally we have the following result [Kar00, Theorem 1.8].

Theorem. *Let G be a finite group and V a real representation of G. Then $K^0_G(V)$ (resp. $K^1_G(V)$) is free abelian of rank equal to the number of oriented even (resp. odd) conjugate classes in G.*

8. The case of symmetric groups

Let S_n denote the symmetric group. Assume that $\mathbb{R}^n$ has a linear action of S_n by permuting the coordinates. Denote by p_n (resp i_n) the number of partitions $(\lambda_1, \ldots, \lambda_k)$ of n such that $1 \leq \lambda_1 < \cdots < \lambda_k$ and k is even (resp. odd). Karoubi mentioned in [Kar00] that the following result was proved by Le Gall and Monthubert.

Theorem. *The rank of $K^0_{S_n}(\mathbb{R}^n)$ and $K^1_{S_n}(\mathbb{R}^n)$ are p_n and i_n, respectively.*

It is interesting to note that

(1) $p_n - i_n$ is the coefficient of q^n in the theta function
$$\prod_{m=1}^{\infty}(1-q^m) = \sum_{m=-\infty}^{\infty}(-1)^m q^{\frac{m(3m-1)}{2}}.$$
In particular, $p_n - i_n = (-1)^m$ if $n = \frac{m(3m-1)}{2}$ and 0 otherwise.

(2) $p_n + i_n$ is asymptotically equivalent to
$$\frac{e^{\pi\sqrt{\frac{n}{3}}}}{4 \cdot 3^{\frac{1}{4}} \cdot n^{\frac{3}{4}}}.$$

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