Trends in Mathematics
Information Center for Mathematical Sciences
Volume 4, Number 1, June 2001, Pages 80–83

A GENERALIZED MEAN VALUE PROPERTY

JAESUNG LEE

ABSTRACT. If a bounded function f on the complex bidisc D^2 satisfies the conformally invariant mean value property with respect to any two radial probability measures on the unit disc, then f is harmonic with respect to each variable.

Let D be the open unit disc of $\mathbb{C}$ and m be the Lebesgue measure normalized to $m(D) = 1$. If $u \in L^1(D, m)$ is harmonic, then u satisfies an invariant volume mean value property ; that is

$$u\big(\ \varphi(0)\ \big) \ = \ \int_D u \circ \varphi \ dm \quad \text{for every } \varphi \ \in \ Aut(D) \tag{1}$$

which is equavalent to

$$u(z) \ = \ \int_D u\big(\ \varphi_z(x)\ \big)\ dm(x) \quad \text{for all} \ \ z \in D \quad \text{where } \varphi_z(x) \ = \ \frac{z-x}{1-\bar{z}x}$$

One of the result of [1] is that if $u \in L^1(D, m)$ satisfies (1), then u is harmonic. But the outcome is different when we replace D by D^n.
Indeed [3] showed that when $1 \le p < \infty$, $f \in L^p(D^2, m \times m)$ satisfying

$$f\big(\ \psi(0,0)\ \big) \ = \ \int\int_{D^2} f \circ \psi \ dmdm \quad \text{for every } \psi \in Aut(D^2)$$

doesn't have to be 2-harmonic.
However, if $f \in L^\infty(D^n)$ satisfies

$$f\big(\ \psi(0,\cdots,0)\ \big) \ = \ \int_D \cdots \int_D f \circ \psi \ dm \cdots dm \quad \text{for every} \ \ \psi \in Aut(D^n)$$

then f is $n-$ harmonic (which means $\Delta_1 f = \cdots = \Delta_n f = 0$).
On the other hand, by using the polar coordinate we can observe that if f is $n-$ harmonic and $\tau_1, \cdots, \tau_n$ are radial probability measures on D, then

$$f\big(\ \psi(0,\cdots,0)\ \big) \ = \ \int_D \cdots \int_D f \circ \psi \ d\tau_1 \cdots d\tau_n \quad \text{for every } \psi \in Aut(D^n) \tag{2}$$

And from the structure of the automorphism of the polydisc (p.167 of [4]) and the rotation invariance of $\tau_1, \cdots, \tau_n$, it follows that (2) is equivalent to

$$f(z_1, \cdots, z_n) \ = \ \int_D \cdots \int_D f\big(\ \varphi_{z_1}(x_1), \cdots, \varphi_{z_n}(x_n)\ \big)\ d\tau_1(x_1) \cdots d\tau_n(x_n) \tag{3}$$

2000 *Mathematics Subject Classification.* Primary 47A15; Secondary 46C15.
Key words and phrases. mean value property, Banach algebra, n-harmonic function.
Received August 30, 2000.

Here we extend the result of [3] by showing that $f \in L^\infty(D^n)$ satisfying (3) is also $n-$ harmonic. Before that we need some preliminaries.

Let μ be the conformally invariant measure on D defined by

$$d\mu(z) = (1-|z|^2)^{-2}\ dm(z)$$

and let $L^p_R(\mu)$ be the subspace of $L^p(\mu)$ which consists of radial functions.
It is known that $L^1_R(\mu)$ is a commutative Banach algebra under the convolution

$$(u * v)(z) \ = \ \int_D u\big(\ \varphi_z(x)\big)\ v(x)\ d\mu(x)$$

Likewise if τ is a radial measure we write

$$(u * \tau)(z) \ = \ \int_D u\big(\ \varphi_z(x)\big)\ d\tau(x)$$

Here we use the folllowing theorems of Benyamini and Weit.

Theorem 2.1 **of** [2] Let $\tau(\neq \delta_0)$ be a radial probability measure on D. If $u \in L^1_R(\mu)$ satisfies

$$\int_D u\ d\mu \ = \ 0$$

then $u * \tau^n \ \rightarrow \ 0$ in the norm of $L^1(\mu)$.

Theorem 3.1 **of** [2] Let $\tau(\neq \delta_0)$ be a radial probability measure on D. If $v \in L^\infty(D)$ satisfies $v * \tau = v$ then v is harmonic.

Next theorem is our main result. Eventhough we restrict ourselves to the case of $n = 2$, the same method with induction should hold for any dimensional polydisc.

Theorem 1. *Let $\tau_1, \tau_2(\neq \delta_0)$ be radial probability measures on D. If $f \in L^\infty(D^2)$ satisfies*

$$f(z,w) \ = \ \int\int_{D^2} f\big(\ \varphi_z(x), \varphi_w(y)\ \big)\ d\tau_1(x) d\tau_2(y) \quad \textit{for all} \ \ z,w \in D$$

then f is $2-$ harmonic.

Proof. Let's denote

$$(Tf)(z,w) \ = \ \int\int_{D^2} f\big(\ \varphi_z(x), \varphi_w(y)\ \big)\ d\tau_1(x) d\tau_2(y)$$

and assume $f \in L^\infty(D^2)$ satisfies $Tf = f$.
First we prove the case that f is radial, *i.e* $f(z,w) = f(|z|,|w|)$ for all $z,w \in D$.
Since $\tau_1(D) = \tau_2(D)$ and τ_1, τ_2 are positive measures, for $v \in L^\infty_R(D)$ we can write by induction,

$$(v * \tau_1^n)(z) \ = \ \int_D v(x)\ P_n(z,x)\ d\tau_1(x)$$

and

$$(v * \tau_2^n)(z) \ = \ \int_D v(x)\ Q_n(z,x)\ d\tau_2(x)$$

for some $P_n(z,x) > 0, Q_n(z,x) > 0$ which satisfy

$$\int_D P_n(z,x)\ d\tau_1(x) \ = \ \int_D Q_n(z,x)\ d\tau_2(x) \ = \ 1 \qquad \text{for all} \quad n \geq 1.$$

Hence we get

$$\| \ v * \tau_1^n \ \|_\infty \ \leq \ \| \ v \ \|_\infty$$

and

$$\| \ v * \tau_2^n \ \|_\infty \ \leq \ \| \ v \ \|_\infty.$$

Now fix $w \in D$ and define $(f_w)_n \in L_R^\infty(D)$ as

$$(f_w)_n(z) \ = \ \int_D f(z,y)\ Q_n(w,y)\ d\tau_2(y).$$

Then

$$\| \ (f_w)_n \ \|_\infty \ \leq \ \| \ f \ \|_\infty.$$

And

$$\begin{aligned} \big(\ (f_w)_n * \tau_1^n \ \big) \ &= \ \int_D (f_w)_n(x)\ P_n(z,x)\ d\tau_1(x) \\ &= \ \int\int_{D^2} f(x,y)\ P_n(z,x)\ Q_n(z,y)\ d\tau_1(x)\ d\tau_2(y) \\ &= \ (T^n f)(z,w) \end{aligned}$$

Now let $u \in L_R^1(\mu)$ satisfy

$$\int_D u\ d\mu \ = \ 0.$$

Then for fixed $w \in D$, we have

$$\begin{aligned} \int_D u(z)\ f(z,w)\ d\mu(z) \ &= \ \int_D u(z)(T^n f)(z,w)\ d\mu(z) \\ &= \ \int_D u(z)\big(\ (f_w)_n * \tau_1^n \ \big)\ d\mu(z) \\ &= \ \big(\ u * (f_w)_n * \tau_1^n \ \big)(0) \\ &= \ \big(\ (u * \tau_1^n) * (f_w)_n \ \big)(0) \\ &\quad \text{(The convolution is commutative and associative} \\ &\quad \text{since } u, \tau, (f_w)_n \text{ are radial)} \\ &= \ \int_D (u * \tau_1^n)(z)\ (f_w)_n(z)\ d\mu(z) \end{aligned}$$

Hence

$$\left| \int_D u(z)\ f(z,w)\ d\mu(z) \right| \ \leq \ \| \ (f_w)_n \ \|_\infty \ \| \ u * \tau_1^n \ \|_{L^1(\tau)}.$$

But

$$\| \ (f_w)_n \ \|_\infty \ \leq \ \| \ f \ \|_\infty$$

and by Theorem 2.1 of [2], we get

$$\| \ u * \tau_1^n \ \|_{L^1(\tau)} \ \to \ 0 \qquad \text{as} \qquad n \to \infty.$$

Hence we conclude that

$$\int_D u(z)\ f(z,w)\ d\mu(z) \ = \ 0$$

which means $f(z,w)$ is a constant for every fixed w, *i.e* , f is a function of w variable only.
Let $f(z,w) = g(w)$, then $Tf = f$ implies that $g * \tau_2 = g$. Now applying Theorem 3.1 of [2] , we conclude that g is a constant (since g is radial and harmonic). Hence f is a constant.
Now we go to the general case. Let $f \in L^\infty(D^2)$ satisfies $Tf = f$. Then we consider the radialization Rf of f defined by

$$(Rf)(z,w) \;=\; \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} f(ze^{i\theta}, we^{i\eta})\; d\theta\; d\eta.$$

Sinec both R and T are contractions on $L^\infty(D^2)$, by Fubini,

$$T(Rf) \;=\; R(Tf) \;=\; Rf.$$

Hence Rf is a constant , which means

$$f(0,0) \;=\; \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} f(ze^{i\theta}, we^{i\eta})\; d\theta\; d\eta \qquad \text{for all } z,w\in D \tag{1}$$

Now pick $(z,w) \in D^2$ and let $\psi \in Aut(D^2)$ be defined by $\psi(x,y) \;=\; \big(\,\varphi_z(x), \varphi_w(y)\,\big)$.
Then by the rotation invariance of τ_1 and τ_2, we can easily get

$$T(f\circ\psi) \;=\; Tf\circ\psi \;=\; f\circ\psi.$$

hence we can replace f by $f\circ\psi$ in (1) to get

$$f(z,w) \;=\; \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} f\big(\,\varphi_z(xe^{i\theta}), \varphi_w(ye^{i\eta})\,\big)\; d\theta\; d\eta \qquad \text{for any } x,y \;\in\; D$$

Put $y = 0$ in the above equation then use 4.2.4 of [5] ($Tf = f$ means $f \in C^\infty(D^2)$) to get $\Delta_1 f = 0$, then we put $x = 0$ to get $\Delta_2 f = 0$.
Hence f is $2-$ harmonic , which completes the proof of the theorem. □

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School of Mathematics, Korea Institute for Advanced Study, Seoul 130-012, Korea
E-mail address: jalee@kias.re.kr