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ON FLOER COHOMOLOGY

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ABSTRACT. In this article, we briefly survey on the recent researches related to Floer Cohomogy.

In 1980's, Floer developed new (Co-)homology theory to prove the Arnold conjecture:

Conjecture 1. *Let (M^{2n}, ω) be a symplectic manifold, i.e, ω is a nondegenerate closed 2-form. For any hamiltonian diffeomorphism ϕ, if all the fixed points of ϕ are nondegenerate, then*

$$\#\{\textit{fixed points of } \phi\} \geq Crit(M).$$

I will briefly explain definitions. A hamiltonian diffeomorphism is a diffeomorphism of M such that for corresponding vector field X, $i(X)\omega$ is exact. A fixed point x of ϕ is called nondegenerate if the graph of ϕ intersect the diagonal in $M \times M$ transversally at (x, x). $Crit(M)$ is the minimal number of critical points for a Morse function on M.

This conjecture can be stated more generally for lagrangian intersection problem.

Conjecture 2. *Let $L \subset (M, \omega)$ be a lagrangian submanifold, i.e,*

$$\omega|_{T_x L} \equiv 0$$

for all $x \in L$. If the intersection of L and $\phi(L)$ are transveral, then

$$\#(L \cap \phi(L)) \geq Crit(L).$$

Conjecture 1 follows from Conjecture 2. Here we only consider transveral cases.

In general, it is difficult to calculate $Crit(L)$. But from the standard Morse theory, we know

$$Crit(L) \geq SB(M) := \sum_{i=0}^{\dim L} b_i(L)$$

where $b_i(L)$ is the betti number of L. Therefore, people prove the conjectures replacing $Crit(L)$ by $SB(M)$. Conjecture 1 is proven by many people (Fukaya-Ono, Ruan, Tian, Seibert,...) and Conjecture 2 is proven recently.

The construction of Floer Cohomology is the extension of finite dimensional Morse theory to an infinite dimensional path space.

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Here we present a quick review of the construction of Floer cohomology. Define the space

$$\Omega := \Omega(L,\phi) = \{z \in C^\infty([0,1],M) | z(0) \in L, z(1) \in \phi(L) \text{ and } [\phi_t(z(t))] = 0 \text{ in } \pi_1(M,L)\}.$$

The tangent space $T_z\Omega$ consists of tangent fields ξ of M along z which are tangent to L at 0 and to $L' := \phi(L)$ at 1. Then ω induces a closed 1-form

$$\alpha(\xi) = \int_0^1 \omega(\dot{z}(t), \xi(t))dt$$

on Ω. Under suitable hypothesis, e.g, $\pi_2(M,L) = 0$, the form can be integrated to a globally defined function $\mathcal{A}$ such that

$$\alpha(z) = d\mathcal{A}(z).$$

The function $\mathcal{A}$ play role of a Morse function in finite dimension Morse theory. However, generally, the function $\mathcal{A}$ defined only locally, i.e, it is a circle valued function. It is easily checked that the critical points of $\mathcal{A}$ is the intersection points of L and L'.

Now, we should find the flows connecting critical points. For this, choose a compatible almost complex structure J. The connecting two critical points x, z is the smooth map $u(\tau, t) : \mathbb{R} \times [0,1] \to M$ such that

(i)
$$\bar{\partial} u := \frac{\partial u}{\partial \tau} + J\frac{\partial u}{\partial t} = 0$$

(ii)
$$\lim_{\tau \to -\infty} u(\tau) = x, \qquad \lim_{\tau \to \infty} u(\tau) = z.$$

Denote $\mathcal{M}(x,z)$ be the moduli space of such trajectories and $\widehat{\mathcal{M}}(x,z)$ be the quotient moduli space by time translation.

Unlike finite dimensional Morse theory, we can assign index to each critical points naturally. But Maslov-type index can be given for each critical points. Set $\mu(x)$ the index of x. The dimension of the moduli space $\mathcal{M}(x,z)$ is given by $\mu(x) - \mu(z)$.

Let C^k be free $\mathbb{Z}_2$ module generated by critical points of index k. Define a coboundary operator $\delta : C^k \to C^{k+1}$ such that for $x \in C^k$ and $y \in C^{k+1}$

$$\langle \delta x, y \rangle = \#\mathcal{M}(y,x) \mod 2.$$

We see that $\delta \circ \delta = 0$ and this defines a cohomology $FH(M,L,\phi)$. One of important steps is to prove

Theorem 3. *The Cohomology $FH(M,L,\phi)$ does not depend on ϕ.*

With this theoem, if ϕ is small, then we can find a Morse function $F : L \to \mathbb{R}$. Hence, it is the just finte dimensional Morse theory.

Remark 4. Floer (co)homology is not defined for all symplectic manifolds. It is due to the bigger dimension of bubbles. Hence the proof of Arnold conjecture was done by using moduli space of stable maps and Gromov-Witten invariants in the case of Conjecture 1. In the case of Conjecture 2, Fukaya, Ono, Oh and etc extended the Floer Homology theory to A^∞-Category and the proof was done with it. Although, Floer (co)homolgy gives a good invariant in the case it could be defined. There are many results by calculating Floer homolgy groups.

Originally, to prove Theorem 3, Floer investigated the change of Moduli spaces when there comes non-transeversal intersection in the isotopy. Afterwards, the proof is altered to avoid Floer's complicated arguemnts. His proposition is as follows: Let ϕ_λ be the deformation such that L and $\phi_0(L)$ has a non-transversal intersection y. Let U be a small neighborhood of y. Under such a deformation, we assume that for $\lambda < 0$ small enough, there exists a pair of transverse intersection y^+_λ, y^-_λ in U, whereas for $\lambda > 0$, U does not contain any intersections at all.

Proposition 5. *Let $x, z \in L \cap \phi_0(L)$ be the transverse intersections with $\mu(x) = \mu(y^+)$ and $\mu(z) = \mu(y^-)$. Then there exists J and $\epsilon > 0$ so that we have bijections between finite sets*

$$\widehat{\mathcal{M}}_\epsilon(x,z) \simeq \widehat{\mathcal{M}}_{-\epsilon}(x,z) \cup \big(\widehat{\mathcal{M}}_{-\epsilon}(x,y^-_{-\epsilon}) \times \widehat{\mathcal{M}}_{-\epsilon}(y^+_{-\epsilon},z)\big).$$

The proof of this proposition is very complicated to understand. Hence, many people made a new proof of the isotropy invariance not using this proposition. But, in the applications of Floer Cohomology, sometimes, it is neccesary to establish a proposition analogue Proposition 5.

Note that this proposition has a strong assumption $\pi_2(M,L) = 0$. But, with weaker assumption, the function $\mathcal{A}$ is defined only locally. In this case, the moduli space of trajectories connecting two critical points could be nonempty. Hence we should modify Propostion 5. Let $\widehat{\mathcal{M}}(y)$ be the moduli space of trajectories starting from y and ending at y. Then, we have the following proposition.

Proposition 6. *Let $x, z \in L \cap \phi_0(L)$ be the transverse intersections with $\mu(x) = \mu(y^+)$ and $\mu(z) = \mu(y^-)$. Then there exists J and $\epsilon > 0$ so that we have bijections between finite sets*

$$\widehat{\mathcal{M}}_\epsilon(x,z) \simeq \widehat{\mathcal{M}}_{-\epsilon}(x,z) \cup \big(\cup_k \widehat{\mathcal{M}}_{-\epsilon}(x,y^-_{-\epsilon}) \times (\times^k \widehat{\mathcal{M}}(y)) \times \widehat{\mathcal{M}}_{-\epsilon}(y^+_{-\epsilon},z)\big).$$

The differences of the proof of this proposition are dealing with new moduli space $\widehat{\mathcal{M}}(y)$ and constructing an approximte solution. Since the story is too long to describe, we omit it here.

Remark 7. This proposition is useful in the construction of an invariant counting trajectories, since this proposition shows the change of numbers.

Another development of Floer Cohomogy is to define it for other sets rather than lagrangian submanifolds. Consider contangent bundle $T*M$ for any smooth manifold M. For any smooth submanifold S of M, consider conormal bundle v^*S in T^*M. Then the conormal bundle is a Lagrangian submanifold of T^*M. In this case, Oh proved the following theorem.

Theorem 8. *In the above situation,*

$$\#(\phi(o_M) \cap v^*S) \geq SB(S).$$

Oh and Kasturirangan [KaO] extended this theorem for open sets with smooth boundary ∂U in M. To obtain a smooth Lagrangian submanifold in T^*M, they considered the cornormal set $\nu^*\overline{U}$

$$\nu^*\overline{U} := o_U \coprod \nu^*_-(\partial U)$$

where $\nu^*_-(\partial U)$ is the negative part of the conormal bundle of ∂U. Since $\nu^*\overline{U}$ is not a smooth set, they consider a Lagrangian approximation by smoothing Lipshitz corners of $\nu^*_-(\partial U)$. And then, they defined the Floer Homology by the direct limit of Floer homologies defined by Lagrangian approximations.

One can consider the generalization of this theorem to more general sets in M.

In other direction, recently, some people are interested in possibility of defining Floer cohomology theory for singular Lagrangian submanifolds. To define such theory, one may consider singularity resolution, i.e, symplectic blowing up. Let $M = \mathbb{C}^n$ and L be a singular Lagrangian submanifold of M. Let $\pi : \tilde{M} \to M$ be a symplectic blowing up. Unfortunately, $\pi^{-1}(L)$ is not Larangian in $\widetilde{M}$. Hence this job fails. Nevetheless, since this problem has connection with the problem of (singular) Lagrangian embedding problem, the study of Floer cohomology theory of singular lagrangian subamnifold is important.

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