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BRUHAT-TITS BUILDING OF A p-ADIC REDUCTIVE GROUP

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ABSTRACT. A Bruhat-Tits building of a p-adic reductive group is a p-adic analogue of the global Riemannian symmetric space of a semisimple Lie group. It is useful in studying structure of G and its various compact open subgroups as well as its representations. This note is a short introduction to the theory starting from the Tits system.

Let G be a semisimple Lie group and K be a maximal compact subgroup of G which is the set of fixed poiints under a Cartan involution. Then G/K is a global Riemannian symmetric space. For example, $\mathrm{SL}_2(\mathbf{R})/\mathrm{SO}_2(\mathbf{R})$ is the Poincaré upper half plane. When G is a semisimple p-adic group, its maximal compact subgroup K is always open (for example, $\mathrm{SL}_n(\mathbf{Z}_p) \subset \mathrm{SL}_n(\mathbf{Q}_p)$) hence the coset space G/K is a discrete set. The Bruhat-Tits building $\mathfrak{B}$ of G is a complete metric space having a (poly)simplicial complex structure with isometric and simplicial G-action. The set G/K is a subset of vertices of $\mathfrak{B}$ and the G-action on these vertices is the same as the translations on G/K. This note is a short introduction to Bruhat-Tits buildings of p-adic reductive groups, which were defined and throughly investigated in [1, 2].

A *Tits system* is a quadruple (G, B, N, S) consisting of a group G and its subgroups B, N and a subset S of $W = N/B \cap N$ satisfying the following axioms.
(T1) $B \cup N$ generates G and $B \cap N$ is a normal subgroup of N.
(T2) S generates W and its elements are of order 2.
(T3) For any $s \in S$ and $w \in W$, we have $sBw \subset BwB \cup BswB$.
(T4) For any $s \in S$, we have $sBs \neq B$.

The group W is called the Weyl group of the Tits system. The group G is sometimes called a group with a BN-pair.

Example 1. Let $G = \mathrm{SL}_n(k)$ where k is a field, B (resp. T) be the subgroup of G of upper triangular (resp. diagonal) matrices, and N be the normalizer of T in G. Then the Weyl group W is isomorphic to the symmetric group S_n of n letters. Let $S = \{(1\ 2), (2\ 3), \ldots, (n-1\ n)\}$ under this isomorphism then (G, B, N, S) is a Tits system.

Let (G, B, N, S) be a Tits system, then G is the disjoint union of double cosets BwB for $w \in W$.
(*Bruhat Decomposition*) $\qquad G = BWB$.

A *parabolic subgroup* of G is a subgroup containing a conjugate of B. Let X be a subset of S and W_X be the subgroup of W generated by X. Then $B_X = BW_XB = \cup_{w \in W_X} BwB$ is a parabolic subgroup and conversely any parabolic subgroup of G containing B is of this form. Moreover we have $B_X \cap B_Y = B_{X \cap Y}$. Hence a

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parabolic subgroup is conjugate to one and only one B_X for some $X \subset S$, which is called the *type* of P. It follows from (T4) and the Bruhat decomposition that B is its own normalizer. The same is true for any parabolic subgroup. A proper subgroup P which is parabolic and $B\backslash P/B$ is finite i.e. W_X is finite, is called a *parahoric subgroup.*

We will suppose the Weyl group W is an irreducible Coxeter group in this note. The Bruhat-Tits building $\mathfrak{B}$ of G as a simplicial complex is defined as follows. (In general, W is a product of irreducible Coxeter groups. And the building is the product of buildings corresponding to irreducible Weyl groups.) The set of simplices of $\mathfrak{B}$ is the set of parahoric subgroups of G. Let σ_P denote the simplex corresponding to the parahoric subgroup P. The face relation is given by the inverse inclusion so σ_P is a face of σ_Q if and only if $P \supset Q$. The dimension of $\mathfrak{B}$ is $|S| - 1$ and if P is a parahoric subgroup of type X then the codimension of σ_P is $|X|$. The group G acts on $\mathfrak{B}$ by conjugation on parahoric subgroups, which is obviously a simplicial map.

Simplices of $\mathfrak{B}$ of maximal dimension are called *chambers*, hence they correspond to conjugates of B. The chamber $C_0 = \sigma_B$ corresponding to B is called the fundamental chamber and the subcomplex $A_0 = N \cdot \overline{C_0}$ is called the standard apartment. For any $g \in G$, the subcomplex $g \cdot A_0$ is called an *apartment* of $\mathfrak{B}$. When W is finite, an apartment is homeomorphic to a sphere and $\mathfrak{B}$ is called a *spherical buiding.* When W is infinite i.e. when W is an affine Weyl group, an apartment is homeomorphic to a Euclidean space on which elements of $S \subset W$ act by orthogonal reflections with respect to hyperplanes. $\mathfrak{B}$ is called an *affine* (or *Euclidean*) *building* in this case.

Proposition 1. *(i) G acts transitively on the set of pairs (A, C) of apartments and chambers in them.*

(ii) Given any two simplices, there exists an apartment containing both of them.

(iii) If A, A' are two apartments containing a chamber C, then there exists a unique simplicial isomorphism $\rho : A' \to A$ given by restriction of the action of some $g \in G$ such that ρ is identity on $A \cap A'$.

(iv) Given a pair (A, C) of an apartment and a chamber in it, there exists a unique retraction $\rho_{A,C} : \mathfrak{B} \to A$ centered at C, which are obtained by gluing maps of (iii) above.

So far we have defined a simplicial complex from a Tits system, a group with some special subgroups. This procedure can be reversed, which is the viewpoint of [3]: A chamber complex is a simplicial complex where every simplex is a face of a chamber. Suppose a group G acts on a simplicial complex $\mathfrak{B}$ equipped with a system of chamber subcomplexes, called apartments, in such a way that the first statement of the above proposition holds and if $g \in G$ stabilizes a facet F then g fixes the vertices of F, in other words, the action of G is *type preserving.* The case when we drop the last assumption will be explained at the end of this note. Let A_0 be an apartment and C_0 be a chamber in it. Let B and N be stabilizers of C_0 and A_0, respectively. Then (G, B, N, S) is a Tits system where $S \subset W = N/B \cap N$ is the set of simple reflections corresponding to walls of C_0.

Example 2. Let k be a non-archimedean local field with residue field $\bar{k} = \mathfrak{o}/\mathfrak{p}$, where $\mathfrak{o}$ and $\mathfrak{p}$ denote the ring of integers of k and the maximal ideal of $\mathfrak{o}$, respectively. Let V be an n-dimensional vector space over k and $G = \mathrm{SL}(V)$. We construct the building $\mathfrak{B}$ of G as follows: Vertices of $\mathfrak{B}$ are homothety classes of $\mathfrak{o}$-lattices in V.

The vertices $\lambda_1, \dots, \lambda_t$ form a simplex in $\mathfrak{B}$ if and only if there exist representatives $x_1 \in \lambda_1, \dots, x_t \in \lambda_t$ such that $x_1 \supset \cdots \supset x_t \supset \mathfrak{p}x_1$. G acts on $\mathfrak{B}$ in an obvious way. An apartment of $\mathfrak{B}$ corresoponds to the collection of coordinate lattices with respect to a base of V. More precisely, let $e_1, \dots, e_n$ be a base of V. The set of simplexes having vertices of the form $\mathfrak{p}^{m_1}e_1 + \cdots + \mathfrak{p}^{m_n}e_n$ is an apartment A_0 of $\mathfrak{B}$. And the vertices $x_1 = \mathfrak{o}e_1 + \cdots + \mathfrak{o}e_n \subset x_2 = \mathfrak{p}^{-1}e_1 + \mathfrak{o}e_2 + \cdots \subset \cdots \subset x_n = \mathfrak{p}^{-1}e_1 + \cdots + \mathfrak{p}^{-1}e_{n-1} + \mathfrak{o}e_n \subset \mathfrak{p}^{-1}x_1$ form a chamber C_0 in A_0. It is easy to show the followings: The stabilizer B of C_0 is the set of elements $g \in \mathrm{SL}_2(\mathfrak{o})$ which are congruent to upper triangular matrices modulo $\mathfrak{p}$. The stabilizer N of the apartment A_0 is the normailzer in G of the subgroup T of the diagonal matrices. Hence $B \cap N$ is the diagonal matrices with entries in $\mathfrak{o}^*$ and W is isomorphic to the semidirect product of $\mathbf{Z}^n$ by S_n.

From now on we consider only affine buildings. We fix a Euclidean metric on the standard apartment A_0 which is invariant with respect to the action of N. Transfer this metric to each apartment $A = g \cdot A_0$ and it induces a well-defined metric d on the whole building: Recall that arbitrary two points in $\mathfrak{B}$ is contained in a single apartment.

Theorem 1. *(i) The building $\mathfrak{B}$ is complete with respect to the metric d.*

(ii) Geodesics in $\mathfrak{B}$ are straight lines in apartments.

(iii) (Negative curvature inequality) Let $x, y, z \in \mathfrak{B}$ and m be the midpoint of the segment xy. Then we have

$$d(x,z)^2 + d(y,z)^2 \geq 2d(m,z)^2 + \frac{1}{2}d(x,y)^2 \ .$$

(iv) Let M be a bounded convex subset of $\mathfrak{B}$. Then the stabilizer of M in Isom($\mathfrak{B}$) has a fixed point in the closure of M.

(i) and (iii) can be proved using the retractions $\rho_{A,C}$ in Proposition 1 (iv), which are distance reducing. For (iv), consider the set M_1 of midpoints of pairs of points in M, the set M_2 of midpoints of M_1 and so on. Then using (iii) it can be shown that the diameters of M_i decay exponentially and $x \in \cap \overline{M_i}$ is a desired fixed point.

A subgroup K of G is said to be *bounded* if $B\backslash K/B$ is finite. If G is a topological group such that B is open compact, for exmaple as in the Example 2 above, then a subgroup is bounded if and only if it is compact.

Corollary 1 (Fixed point theorem)**.** *Any bounded subgroup of G has a fixed point in $\mathfrak{B}$.*

Indeed, if K is a bounded subgroup, then the convex hull of $K \cdot \overline{C_0}$ is a K-stable bounded convex subset of $\mathfrak{B}$. It follows that a maximal bounded subgroup fixes a vertex of $\mathfrak{B}$, hence a maximal bounded subgroup is a maximal parahoric subgroup and vice versa.

A *vector chamber* is a connected component of the complement in an apartment A of hyperplanes which are kernels of "vector parts" of walls. And a *sector* is a translation $x+D$ of a vector chamber D by $x \in A$. The intersection $(x+D)\cap(y+D)$ of sectors with the same vector part D is again a sector $z + D$. If we define two sectors are equivalent if their intersection is again a sector, then the equilvalence classes of sectors define chambers of some simplicial complex which is called the spherical building at the infinity of $\mathfrak{B}$. In this way we can compactify the affine building $\mathfrak{B}$ by adding a spherical building.

Let D be a vector chamber in the standard apartment A_0 and P_{x+D} be the stabilizer of the sector $x + D$. Their union $\mathcal{B}^0 = \cup_{x \in A_0} P_{x+D}$ is a subgroup of G since $P_{x+D} \cup P_{y+D} \subset P_{(x+D)\cap(y+D)}$. Then we have
(*Iwasawa decomposition*) $\qquad G = \mathcal{B}^0 W B$.

This follows from the fact that for any pair of a sector $\mathcal{S}$ and a chamber C there exists an apartment containing C and a subsector of $\mathcal{S}$.

Example 3. Let $G = \mathrm{SL}_n(k)$ with k a non-archimedean local field as in the previous example. Then the set of upper triangular matrices in G with diagonal entries in $\mathfrak{o}^*$ is $\mathcal{B}^0$ for a suitable vector chamber D.

A vertex x in an apartment A is called *special* if for any wall in A there exists a wall parallel to it and passing through x. Clearly, every chamber has at least one special vertex. The stabilizer of a special vertex in A is called a *good* maximal bounded subgroup of G (with respect to A). For example $\mathrm{SL}_n(\mathbf{Z}_p) \subset \mathrm{SL}_n(\mathbf{Q}_p)$ is a good maximal bounded subgroup. Let K be a good maximal bounded subgroup containing B and $X \subset S$ be the type of K. Let V be the subgroup of translations in W. Then W is the semidirect product of V by W_X and we have
(*Iwasawa decomposition'*) $\qquad G = \mathcal{B}^0 V K$
(*Cartan decomposition*) $\qquad G = K V_D K$
where V_D denotes the set of translations in W which map D into itself.

The Euclidean buildings we have discussed so far are those of simply connected semisimple p-adic groups. To deal with a general semisimple or reductive p-adic group we proceed as follows. Let (G, B, N, S) be a Tits system. A group homomorphism $\phi : G \to \hat{G}$ is said to be *BN-adapted* if
(1) the kernel of ϕ is contained in B and
(2) for any $g \in \hat{G}$ there exists $h \in G$ such that $\phi(hBh^{-1}) = g\phi(B)g^{-1}$ and $\phi(hNh^{-1}) = g\phi(N)g^{-1}$.

Example 4. Let (G, B, N, S) with $G = \mathrm{SL}_n(k)$ be the Tits system of Example 2. Then the natural maps $\mathrm{SL}_n(k) \to \mathrm{GL}_n(k)$ and $\mathrm{SL}_n(k) \to \mathrm{PGL}_n(k)$ are BN-adapted.

Let $\phi : G \to \hat{G}$ be a BN-adapted homomorphism. $\hat{G}$ acts on the set of parabolic subgroups of G by $P \mapsto {}^gP := \phi^{-1}(g\phi(P)g^{-1})$. Hence $\hat{G}$ acts on the building $\mathfrak{B}$ of G. This action is not necessarily type-preserving. In fact $\hat{G}$ acts by automorphisms on the Coxeter group (W, S): For $g \in \hat{G}$ let $h \in G$ be such that the condition (2) above holds. Then for $w \in W$, $\xi(g)w$ is the element of W such that $\phi(hB\xi(g)(w)Bh^{-1}) = g\phi(BwB)g^{-1}$. If P is a parabolic subgroup of type $X \subset S$, then ${}^gP$ is a parabolic subgroup of type $\xi(g)(X)$. Let $\hat{G}_0$ be the kernel of ξ, in other words, $\hat{G}_0$ is the subgroup of elements of $\hat{G}$ which are type-preserving. Let $\hat{B} = \hat{G}_0 \cap \mathrm{Stab}_{\hat{G}} B$. Then $(\hat{G}_0, \hat{B}, \phi(N), S)$ is a Tits system whose Weyl group is canonically isomorphic to that of (G, B, N, S). For a parabolic subgroup P of G let $\hat{P} = \hat{G}_0 \cap \mathrm{Stab}_{\hat{G}} P$. Then $P \mapsto \hat{P}$ is a bijection between parabolic subgroups of G and $\hat{G}_0$. Hence the building of $(\hat{G}_0, \hat{B}, \phi(N), S)$ is isomorphic to $\mathfrak{B}$. The same argument applies when we are given a building $\mathfrak{B}$ and a group $\hat{G}$ acting on it transitively on the set of pairs (A, C) of an apartment and a chamber in it.

The group $\hat{G}$ also has similar decompositions as G. Let $\hat{N}$ be the stabilizer of the fundamental apartment A_0 in $\hat{G}$. So elements of $\hat{N}$ define automorphisms of the affine space A_0. Let $\hat{W}$ be the image of $\hat{N}$ in $\mathrm{Aut}(A_0)$. Then W can be identified with the subgroup of $\hat{W}$ consisting of elements which are type preserving. Let Ξ be the stabilizer of the fundamental chamber C_0 in $\hat{W}$. Hence Ξ is a subgroup of permutations of vertices of C_0. Then $\hat{W}$ is the semidirect product of W by Ξ. Moreover $\hat{G}/\hat{G}_0 \cong \Xi$. Let $\hat{V}$ be the subgroup of translations in $\hat{W}$. Fix a vector chamber D in A_0 and let $\hat{P}_{x+D}$ be the stabilizer in $\hat{G}$ of the vector chamber $x + D$. Then $\hat{\mathcal{B}}^0 = \cup_{x \in A_0} \hat{P}_{x+D}$ is a subgroup of $\hat{G}$ such that $\mathcal{B}_0 = \phi^{-1}(\hat{\mathcal{B}}^0)$. Let K be the stabilizer in $\hat{G}$ of a special vertex in C_0, hence it is a good maximal bounded subgroup containing $\hat{B}$. The following decompositions of $\hat{G}$ follow more or less from those of $\hat{G}_0$.

(*Bruhat decomposition*) $\hat{G} = \hat{B}\hat{W}\hat{B}$

(*Iwasawa decomposition*) $\hat{G} = \hat{\mathcal{B}}^0\hat{W}\hat{B} = \hat{\mathcal{B}}^0\hat{V}K$

(*Cartan decomposition*) $\hat{G} = K\hat{V}_D K$.

Although this axiomatic approach starting from a Tits system enables one to prove many results on buildings, another approach is required for more detailed study of structure of buildings and parahoric subgroups as well as for the very existence of a (generalized) Tits sytem for a general p-adic reductive group. This is done through the *valuated root datum* (or *données radicielles valuées* in French) for which we refer the original monographs [1, 2].

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