Trends in Mathematics
Information Center for Mathematical Sciences
Volume 4, Number 1, June 2001, Pages 66–70

# SEMILINEAR PARTIAL DIFFERENTIAL EQUATIONS ARISING IN THE SELF-DUAL (2+1) DIMENSIONAL CHERN-SIMONS GAUGE THEORIES

DONGHO CHAE

ABSTRACT. In this note we review briefly the theories of multivortex solutions of the self-duality equations (the Bogomol'nyi equations) of (2+1) dimensional Chern-Simons gauge theories with various type of boundary condition near infinity.

## 1. INTRODUCTION

Since the pioneering work of the theoretical physicists Landau and Ginbzburg the low dimensional gauge theories and their multivortex solutions have been studied extensively by physicists(See [12] and references therein), and mathematicians(See [16] and references therein). In particular, Jaffe-Taubes initiated rigorous study of the existence theories as well as the other qualitative behaviors of the solutions of the Abelian-Higgs model[16]. Their method of reduction from the first order Bogoml'nyi system into a semilinear elliptic partial differential equations is now standard, and universal in almost all of (2+1) dimensional self-dual gauge theories. Thanks to this reduction procedure the study of (2+1) dimensional self-dual gauge theories is equivalent to that of semilinear elliptic systems in the plane. Interestingly enough the the reduced partial differential equations have nonlinearities of critical type, and there is no well established theories about them. This fact makes the area more fascinating for mathematicians. Moreover, the resulting equations have strong similarities to those appearing in the prescribed curvature problem in two dimensional conformal geometry studied extensively Kazdan and Warner, and Nirenberg. In this review note we introduce basic facts in this area, and in particular recent results on the Chern-Simons type of gauge theories developed by the author and his colleagues/students.

## 2. PURE CHERN-SIMONS THEORY

In this section we illustrate the procedure from the self-dual Lagrangian into a semilinear elliptic system in the plane by taking a simple model of pure Chern-Simons theory. The Lagrangian density of the (2+1)-dimensional relativistic Chern-Simons gauge field theory is given by

$$\mathcal{L} = \frac{\kappa}{4}\varepsilon^{\mu\nu\rho}F_{\mu\nu}A_\rho + (D_\mu\phi)\overline{(D^\mu\phi)} - \frac{1}{\kappa^2}|\phi|^2(1-|\phi|^2)^2, \tag{1}$$

2000 *Mathematics Subject Classification.* 35Qxx.

*Key words and phrases.* semilinear PDEs, self-dual gauge theories.
This research supported partially by GARC-KOSEF, BSRI-MOE, KOSEF(K95070).
Received August 30, 2000

where $A_\mu(\mu = 0, 1, 2)$ is the gauge field on $\mathbb{R}^3$, $F_{\mu\nu} = \frac{\partial}{\partial x^\mu} A_\nu - \frac{\partial}{\partial x^\nu} A_\mu$ is the corresponding curvature tensor, $\phi = \phi_1 + i\phi_2 (i = \sqrt{-1})$ is a complex field on $\mathbb{R}^3$, called the Higgs field, $D_\mu = \frac{\partial}{\partial x^\mu} - iA_\mu$ is the gauge covariant derivative associated with $A_\mu$, $\varepsilon_{\mu\nu\rho}$ is the totally skewsymmetric tensor with $\varepsilon_{012} = 1$, and finally $\kappa > 0$ is the Chern-Simons coupling constant. Our metric on $\mathbb{R}^3$ is $(g_{\mu\nu}) = diag(1, -1, -1)$. This model was suggested by Hong-Kim-Pac[13] and Jackiw-Weinberg[15] to study vortex solutions of the Abelian Higgs model which carry both electric and magnetic charges(See [11] for a general survey of the model). The Gauss equation (variational equation for $A_0$ ) of (1) is given by

$$\kappa F_{12} = -2|\phi|^2 A_0 \,. \tag{2}$$

Using this relation, and by integration by part the static energy corresponding to (1) can be written as ([8],[10])

$$E = \int_{\mathbb{R}^2} \left\{ \frac{\kappa^2}{4} \frac{F_{12}^2}{|\phi|^2} + \sum_{j=1}^{2} |D_j \phi|^2 + \frac{1}{\kappa^2} |\phi|^2 (1 - |\phi|^2)^2 \right\} dx \tag{3-a}$$

$$= \int_{\mathbb{R}^2} \left\{ |(D_1 \pm iD_2)\phi|^2 + \left| \frac{\kappa F_{12}}{2\phi} \pm \frac{1}{\kappa} \bar{\phi}(|\phi|^2 - 1) \right|^2 \right\} dx \pm \int_{\mathbb{R}^2} F_{12} \, dx \,, \tag{3-b}$$

where $+(-)$ sign are chosen if the integral $\int_{\mathbb{R}^2} F_{12} \, dx$ has nonnegative(nonpositive) sign. Below we choose the upper sign. We have thus

$$E \geq | \int_{\mathbb{R}^2} F_{12} \, dx|$$

and the minimum of the energy is saturated if and only if $(\phi, A)$, $A = (A_1, A_2)$ satisfies the self-duality equations, or the Bogomol'nyi equations:

$$(D_1 + iD_2)\phi = 0, \tag{4}$$

$$F_{12} + \frac{2}{\kappa^2} |\phi|^2 (|\phi|^2 - 1) = 0. \tag{5}$$

The system (4) -(5) is equipped with the following natural boundary conditions

$$|\phi(x)| \to \ 1 \ \ as \ \ |x| \to \ \infty \tag{6}$$

or

$$|\phi(x)| \to \ 0 \ \ as \ \ |x| \to \ \infty \tag{7}$$

in order to make the energy (3-a) finite. The solutions $(\phi, A)$ of (4)-(5) satisfying (6) are called topological solutions, while the solutions of (4)-(5) satisfying the boundary condition (7) are called nontopological solutions. Following Jaffe-Taubes [16], we can reduce system (4)-(5) with (6) or (7) to the more simplified form of partial differential equations as follows . We introduce new variable $(u, \theta)$ by

$$\phi = e^{\frac{1}{2}(u+i\theta)}, \ \ \theta = 2\sum_{j=1}^{N} arg(z - z_j) \,, \ \ z = x_1 + ix_2 \in \mathbb{C}^1 = \mathbb{R}^2, \tag{8}$$

where $z_j$, allowing multiplicities, $j = 1, 2, ...., N$ are the zeros, called the centers of the vorticities, of $\phi(z)$. Then, we can rewrite (4) -(5) with (6) or (7) as (Hereafter, we set $\kappa = 2$ for simplicity.)

$$\Delta u = e^u(e^u - 1) + 4\pi \sum_{j=1}^{N} \delta(z - z_j), \tag{9}$$

$$u(x) \ \to \ 0 \ \ as \ \ |x| \to \ \infty \qquad \text{(topological boundary condition)}$$

or

$$u(x) \ \to \ -\infty \ \ as \ \ |x| \to \ \infty \qquad \text{(non-topological boundary condition).} \tag{10}$$

For topological boundary condition Wang[23] proved existence of general multi-vortex solutions, using the variational method similar to Jaffe-Taubes[16]. Later, Spruck-Yang [19] proved existence of topological solutions, using a more constructive iteration method, and generated even shapes of vortices by numerical simulations. (See [2], [21] also for the study of (4)-(5) in a periodic bounded domain.) The non-topological solutions, however, has not been well understood yet compared to the topological ones. In [18] Spruck-Yang proved existence of radially symmetric non-topological solutions which correspond to solutions of (4)-(5) with a single center. In [5] we proved existence of general type of nontopological multivortex solutions, using the perturbation argument combined with the implicit function theorem[25]. Moreover, we establish precise decay estimates near infinity of our solutions. More specifically we proved the following theorem:

**Theorem[5].** *Let $\{z_j\}_{j=1}^N \subset \mathbb{C}^1$ be arbitrarily given. Then, there exists a solution $(\phi, A)$ to (4)-(5)(with $\kappa = 2$), (7) such that the function $\phi(z)$ has the zeros $\{z_j\}_{j=1}^N$ with possible multiplicities, and the pair $(\phi, A)$ make the energy functional (3-a) finite. Moreover, our constructed solutions satisfy: (i) The decay estimates; there exists $\beta > 0$ such that*

$$|\phi|^2 + |F_{12}| + |D_1\phi|^2 + |D_2\phi|^2 = O\left(\frac{1}{|x|^{2N+4+\beta}}\right) \ \ as \ |x| \to \infty \tag{11}$$

*(ii) Flux integral; there exist a sequence of solutions $\phi_i$ and a sequence of positive numbers $\{\gamma_i\}_{i=1}^{\infty}$, $\lim_{i\to+\infty} \gamma_i = 0$ such that*

$$\Phi = \int_{\mathbb{R}^2} F_{12} dx = \int_{\mathbb{R}^2} \frac{2}{\kappa^2} |\phi_i|^2 (1 - |\phi_i|^2)\, dx = 4\pi(N+1) + \gamma_i. \tag{12}$$

## 3. Remarks on the Other Chern-Simons theories

There are numerous other variations of the pure Chern-Simons theories. In this section we announce existence results of the nontopological multivortex solutions in some of the these other models, which is recently obtained. The Generalzied Chern-Simons model has been introduced in [1]. After the Jaffe-Taubes reduction the self-dual equations become

$$(1 - e^u)\Delta u = e^u |\nabla u|^2 - \lambda(1 - e^u)^2 e^u + 4\pi \sum_{j=1}^{N} \delta(z - z_j).$$

The existence of topological solutions is done by Yang[24], while for the nontopological solutions Chae and Imanuvilov recently succeeded in construction of solutions.

For the relativistic Maxwell-Chern-Simons model, introduced in [18], the reduced semilinear system becomes

$$\Delta u = 2q^2(e^u - 1) - 2q\kappa A_0 + 4\pi \sum_{j=1}^{N} \delta(z - z_j)$$
$$\Delta A_0 = \kappa q(1 - e^u) + (\kappa^2 + 2q^2 e^u) A_0$$

The topological solutions and periodic were constructed by Chae-Kim in [7]and [8] respectively(See also[22]). For the Chern-Simons sigma model, introduced in [17], the resulting equations become

$$\Delta u = \lambda \frac{e^u(e^u - 1)}{(1 + ae^u)^3} + 4\pi \sum_{j=1}^{N_1} \delta(z - z_j^1) - 4\pi \sum_{j=1}^{N_2} \delta(z - z_j^2).$$

For this equation periodic solutions were constructed by Chae and Nam[9]. The Maxwell-Chern-Simons sigma model, introduced in [16], is a generalization of the above Chern-Simons-sigma model. The corresponding self-dual equations become

$$\Delta u = 2\kappa q N + 4q \frac{e^u - 1}{e^u + 1} + 4\pi \sum_{j=1}^{N_1} \delta(z - z_j^1) - 4\pi \sum_{j=1}^{N_2} \delta(z - z_j^2)$$
$$\Delta N = \left( k^2 q^2 + \frac{4e^u}{(1 + e^u)^2} \right) N + 2\kappa q^2 \frac{e^u - 1}{e^u + 1}.$$

For this model the periodic solutions have been recently constructed by Chae and Nam[10]. If we couple the pure Chern-Simons theory to the (2+1) dimmensional Einstein gravity, we also abtain the self-dual theory, called the Einstein-Chern-Simons theory, which was modeled by Cangemi-Lee[3] and Clément[11]. The self-dual equation is reduced to

$$\Delta(\eta + ae^u) = -\lambda a e^{\eta + u}(ae^{2u} - 2(a+1)e^u + 2)$$
$$\Delta u = -\lambda e^{\eta + u}(ae^{2u} - (3a+2)e^u + 2(a+1)) + 4\pi \sum_{i=1}^{N} \delta(z - z_i).$$

Both the topological and nontopological solutions were constructed by Chae-Choe[4].

## References

[1] J. Burzlaff, A. Chakrabarti and D. H. Tchrakian, *Generalized self-dual Chern-Simons vortices*, Phys. Lett. B **293** (1992), 127-131.

[2] L. Caffarelli and Y. Yang, *Vortex condensation in the Chern-Simons-Higgs model:an existence theory*, Comm. Math. Phys. **168** (1995), 321-336.

[3] D. Cangemi and C. Lee, *Self-dual Chern-Simons solitons in (2+1) dimensional Einstein gravity*, Phys. Rev. D **46(10)**, 4768-4771.

[4] D. Chae and K. Choe, *Topological and nontopological multivortex solutions of the Einstein Chern-Simons system*, manuscript in preparation.

[5] D. Chae and O. Yu Imanuvilov, *The existence of non-topological multivortex solutions in the relativistic self-dual Chern-Simons Theory*, submitted.

[6] D. Chae and O. Yu Imanuvilov, *Nontopological multivortex solutions of the generalized self-dual Chern-Simons theory*, manuscript in preparation.

[7] D. Chae and N. Kim, *Topological multivortex solutions of the self-dual Maxwell-Chern-Simons-Higgs System*, J. Diff. Eqns **134 No.1** (1997), 154-182.

[8] D. Chae and N. Kim, *Vortex Condensates in the Relativistic Self-Dual Maxwell-Chern-Simons-Higgs System*, RIM-GARC preprint **97-50**.

[9] D. Chae and H.-S. Nam, *Multiple Existence of the Multivortex Solutions of the Self-Dual Chern-Simons CP(1) Model on a Doubly Periodic Domain*, Lett. Math. Phys. **49** (1999), 297-315.

[10] *On the Condensate Multivortex Solutions of the Self-Dual Maxwell-Chern-Simons CP(1) Model*, submitted.

[11] G. Clement, *Graviting Chern-Simons vortices*, Phys. Rev. D **54(2)** (1996), 1844-1847.

[12] G. Dunne, *"Self-Dual Chern-Simons Theories", Lecture Notes in Physics*, vol. M36, Springer -Verlag, Berlin, New York, 1995.

[13] J. Hong, Y. Kim, and P.Y. Pac, Phys. Rev. Lett. **64** (1990), 2230.

[14] R. Jackiw and S. Y. Pi, Phys. Rev. Lett. **64** (1990), 2969.

[15] R. Jackiw and E. J. Weinberg, Phys. Rev. Lett. **63** (1990), 2234.

[16] A. Jaffe and C. Taubes, *Vortices and Monopoles* (1980), Birkhäuser, Boston.

[17] K. Kimm, K. Lee and T. Lee, *Anyonic Bogomol'nyi Solitons in a Gauged O(3) Sigma Model,*, Phys. Rev. D **53** (1996), 4436-4440..

[18] C. Lee, K. Lee and H. Min,, *Self-Dual Maxwell Chern-Simons Solitons*, Phys. Lett. B **252** (1990), 79-83.

[19] J. Spruck and Y. Yang, *Topological solutions in the self-dual Chern-Simons theory: Existence and approximation*, Ann. Inst. Henri Poincaré **1** (1995), 75-97.

[20] J. Spruck and Y. Yang, *The existence of nontopological solitons in the self-dual Chern-Simons theory*, Comm. Math. Phys. **149** (1992), 361-376.

[21] G. Tarantello, *Multiple condensate solutions for the Chern-Simons-Higgs Theory*, J. Math. Phys. **37** (1996), 3769-3796.

[22] T. Ricciardi and G. Tarantello, *Selfdual vortices in the Maxwell-Chern-Simons-Higgs Theory*, preprint.

[23] R. Wang, *The existence of Chern-Simons Vortices*, Comm. Math. Phys. **137** (1991), 587-597.

[24] Y. Yang, *Chern-Simons solitons and a nonlinear elliptic equation*, Helv. Phys. Acta **71** (1998 (5)), 573–585.

[25] E. Zeidler, *Nonlinear functional analysis and applications, V.1* (1985), Springer-Verlag, New York.


Department of Mathematecs, Seoul National University, Seoul 151-742, KOREA
*E-mail address*: dhchae@math.snu.ac.kr