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GLEASON'S PROBLEM FOR HARMONIC BLOCH FUNCTIONS

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ABSTRACT. On the setting of the half-space $\mathbf{R}^{n-1} \times (0, \infty)$, we investigate some properties of harmonic Bloch functions and then we prove that Gleason's problem for the harmonic Bloch space is solvable.

1. Introduction

For a positive integer $n \geq 2$, let $\mathbf{H} = \mathbf{R}^{n-1} \times (0, \infty)$ denote the upper half-space. By Gleason's problem on $\mathbf{H}$ we mean the following:

Let X be a harmonic function space on $\mathbf{H}$. Given a reference point $a \in \mathbf{H}$, and $u \in X$, do there exist functions $g_1, \ldots, g_n$ in X such that

$$u(z) - u(a) = (z - a) \cdot G(z) \qquad (z \in \mathbf{H}) \tag{1.1}$$

where $G = (g_j)$?

Here, we abuse the notation $z \cdot \zeta = \sum_{j=1}^{n} z_j \zeta_j$ for $z = (z_1, z_2, \ldots, z_n) \in \mathbf{H}$ and $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \in \mathbf{C}^n$. Initiated by Gleason [3] for the ball algebra of the unit ball of $\mathbf{C}^n$, this problem has been studied for various holomorphic function spaces on various domains. See Section 6.6 of [5] for more information in that direction. More recently, Zhu [6] (also see [2]) studied Gleason's problem for holomorphic Bloch space on the unit ball of $\mathbf{C}^n$. In this paper, we solve Gleason's problem for the harmonic Bloch space on $\mathbf{H}$.

For functions $g_1, g_2, \ldots, g_n$ harmonic on $\mathbf{H}$, we say that $G = (g_j)$ is a harmonic *conjugate system* if $\nabla f = G$ for some function f harmonic on $\mathbf{H}$. Here, $\nabla = (\partial/\partial z_j)$ denotes the gradient operator.

A harmonic function u on $\mathbf{H}$ is called a *Bloch* function if

$$\|u\|_{\mathcal{B}} = \sup z_n |\nabla u(z)| < \infty$$

where the supremum is taken over all $z \in \mathbf{H}$. We let $\mathcal{B}$ denote the space of all Bloch functions on $\mathbf{H}$.

The following theorem is the main result of this paper.

Theorem 1.1. *There exists a constant C with the following property: To each $u \in \mathcal{B}$ and $a \in \mathbf{H}$, there corresponds a unique harmonic conjugate system $G = (g_j)$*

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of Bloch functions such that (1.1) *holds and*

$$\sum_{j=1}^{n} ||g_j||_{\mathcal{B}} \le C a_n^{-1} ||u||_{\mathcal{B}}.$$

In Section 2 we review some properties of the reproducing kernel for harmonic Bloch functions. Section 3 is devoted to the proof of Theorem 1.1.

2. Reproducing Kernels

Constants. Throughout the paper we will use the same letter C to denote various constants, often with subscripts indicating dependency, which may change at each occurrence. We will often write $A \lesssim B$ for nonnegative quantities A, B if A is dominated by B times some *inessential* positive constant. Also, we write $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

In this section, we review some preliminary results on reproducing kernel for $\mathcal{B}$. First, we introduce some notations. We will often write a point $z \in \mathbf{H}$ as $z = (z', z_n)$ where $z' = (z_1, \dots, z_{n-1}) \in \mathbf{R}^{n-1}$ and $z_n > 0$. Also, we let $z_0 = (0', 1) = (0, 1)$ throughout the paper. For a function $f \in C^1(\mathbf{H})$, we let $D_j f(z) = \frac{\partial f}{\partial z_j}(z)$ for each j and $z \in \mathbf{H}$. For a function $F \in C^1(\mathbf{H} \times \mathbf{H})$, the ambiguous notation $D_j F(z, w)$ will always mean $D_j[F(\cdot, w)]$ evaluated at $z \in \mathbf{H}$ for each fixed $w \in \mathbf{H}$.

For $z \in \mathbf{H}$ and $w \in \overline{\mathbf{H}}$, set

$$P(z, w) = \frac{2}{n\sigma_n} \cdot \frac{z_n + w_n}{|z - \overline{w}|^n}$$

where σ_n denotes the volume of the unit ball in $\mathbf{R}^n$ and $\overline{w} = (w', -w_n)$. The function P is the so-called extended Poisson kernel for $\mathbf{H}$ which is the harmonic extension of the original Poisson kernel for $\mathbf{H}$. We will use the following property of the extended Poisson kernel:

$$\int_{\mathbf{R}^{n-1}} P(z, w)\, dw' = 1 \tag{2.1}$$

for all $z \in \mathbf{H}$ and $w_n \ge 0$. (See [1] for details and related facts.)

The reproducing kernel $\widetilde{R}$ for $u \in \mathcal{B}$ with $u(z_0) = 0$ is given by

$$\widetilde{R}(z, w) = R(z, w) - R(z_0, w),$$

where

$$\begin{aligned} R(z, w) &= -2 D_n P(z, w) \\ &= \frac{4}{n\sigma_n} \cdot \frac{n(z_n + w_n)^2 - |z - \overline{w}|^2}{|z - \overline{w}|^{n+2}}. \end{aligned} \tag{2.2}$$

That is, we have

$$u(z) = \int_{\mathbf{H}} u(w) \widetilde{R}(z, w)\, dw$$

for all $z \in \mathbf{H}$ and $u \in \mathcal{B}$ with $u(z_0) = 0$. See [1], [4] for details and related topics. A generalized reproducing property of the kernel $\widetilde{R}(z, w)$ is also available [4]:

$$u(z) = -2 \int_{\mathbf{H}} w_n [D_n u(w)] \widetilde{R}(z, w)\, dw$$

for all $z \in \mathbf{H}$ and $u \in \mathcal{B}$ with $u(z_0) = 0$. From this we have

$$u(z) - u(z_0) = -2 \int_{\mathbf{H}} w_n [D_n u(w)] \widetilde{R}(z,w)\, dw \tag{2.3}$$

for all $z \in \mathbf{H}$ and $u \in \mathcal{B}$.

By straightforward calculations using (2.2), it is easily checked that there is a constant C, depending only on n, such that

$$\left| D_k D_j \widetilde{R}(z,w) \right| = |D_k D_j R(z,w)| \le \frac{C}{|z - \overline{w}|^{n+2}} \tag{2.4}$$

for all k, j and $z, w \in \mathbf{H}$.

3. Proof of Theorem 1.1

We begin with a simple fact asserting that, given $a \in \mathbf{H}$ and u harmonic on $\mathbf{H}$, there exists a unique harmonic conjugate system G for which (1.1) holds. In what follows, we let

$$z_t(a) = t(z - a) + a$$

for $a, z \in \mathbf{H}$ and $0 \le t \le 1$. For $a = z_0$, put $z_t = z_t(z_0)$.

Proposition 3.1. *Given $a \in \mathbf{H}$ and u harmonic on $\mathbf{H}$, there exists a unique $T_a u$ harmonic on $\mathbf{H}$ such that $T_a u(a) = 0$ and*

$$u(z) - u(a) = (z - a) \cdot \nabla T_a u(z) \tag{3.1}$$

for all $z \in \mathbf{H}$.

Proof. For the existence, given u harmonic on $\mathbf{H}$, define

$$T_a u(z) = \int_0^1 \left[u\big(z_t(a)\big) - u(a) \right] \frac{dt}{t}$$

for $z \in \mathbf{H}$. Clearly, $T_a u(a) = 0$. Note that the singularity of the integrand at $t = 0$ is removable. Thus, by passing the Laplacian through the integral of the above, we see that $T_a u$ is harmonic on $\mathbf{H}$. Also, we have

$$D_j T_a u(z) = \int_0^1 D_j u\big(z_t(a)\big)\, dt \tag{3.2}$$

for all j and $z \in \mathbf{H}$. It follows that

$$\begin{aligned} u(z) - u(a) &= \int_0^1 \frac{d}{dt} u\big(z_t(a)\big)\, dt \\ &= \sum_{j=1}^n (z_j - a_j) \int_0^1 D_j u\big(z_t(a)\big)\, dt \\ &= (z - a) \cdot \nabla T_a u(z) \end{aligned}$$

for all $z \in \mathbf{H}$.

For the uniqueness, let

$$T_a u(z) = \sum_{k=1}^\infty u_k(z - a)$$

be the homogeneous expansion of $T_a u$ near a. Note

$$(z - a) \cdot \nabla T_a u(z) = \sum_{k=1}^\infty k u_k(z - a)$$

near a. Now, suppose the left side of the above is identically 0. Then, we have $u_k \equiv 0$ for all $k \geq 1$. It follows that $T_a u$ is identically 0 near a and thus on all of $\mathbf{H}$. The proof is complete. □

By Proposition 3.1, given $a \in \mathbf{H}$, we now have an operator T_a acting on the space of all harmonic functions on $\mathbf{H}$. Since T_a has the property (3.1), it is now sufficient to show that each component $D_j T_a$ of the operator ∇T_a has the desired boundedness on $\mathcal{B}$. To do so, we prove a lemma.

Lemma 3.2. *There exists a constant C such that*

$$\int_{\mathbf{H}} \int_0^1 \frac{1}{|z_t(a) - \overline{w}|^{n+2}}\, dt\, dw \leq \frac{C}{a_n z_n}$$

for all $a, z \in \mathbf{H}$.

Proof. Let $a, z \in \mathbf{H}$. Then by (2.1), we have

$$\int_{\mathbf{R}^{n-1}} \int_0^1 \frac{dt\, dw'}{|z_t(a) - \overline{w}|^{n+2}} \lesssim \int_0^1 \frac{dt}{\big[t(z_n - a_n) + a_n + w_n\big]^3}$$

and therefore

$$\begin{aligned} \int_{\mathbf{H}} \int_0^1 \frac{1}{|z_t(a) - \overline{w}|^{n+2}}\, dt\, dw &\lesssim \int_0^1 \int_0^\infty \frac{dw_n\, dt}{\big[t(z_n - a_n) + a_n + w_n\big]^3} \\ &\approx \frac{1}{a_n z_n}. \end{aligned}$$

The proof is complete. □

Now we prove the following which, together with Proposition 3.1, implies Theorem 1.1.

Theorem 3.3. *There exists a constant C such that*

$$\sum_{j=1}^n ||D_j T_a u||_{\mathcal{B}} \leq C a_n^{-1} ||u||_{\mathcal{B}}$$

for all $u \in \mathcal{B}$ and $a \in \mathbf{H}$.

Proof. Fix $a \in \mathbf{H}$. Fix j, k and let $u \in \mathcal{B}$. For $z \in \mathbf{H}$, by differentiating under the integral sign of (3.2), we have

$$D_k D_j T_a u(z) = \int_0^1 t D_k D_j u\big(z_t(a)\big)\, dt. \tag{3.3}$$

Note, by differentiating under the integral sign of (2.3),

$$D_k D_j u(z) = -2 \int_{\mathbf{H}} w_n [D_n u(w)] D_k D_j \widetilde{R}(z, w)\, dw$$

and thus, by (2.4), we have

$$|D_k D_j u(z)| \lesssim \int_{\mathbf{H}} \frac{w_n |D_n u(w)|}{|z - \overline{w}|^{n+2}}\, dw$$

for $z \in \mathbf{H}$. This, together with (3.3), yields

$$\begin{aligned}|\nabla D_j T_a u(z)| &\lesssim \int_{\mathbf{H}} w_n |D_n u(w)| \int_0^1 \frac{t}{|z_t(a) - \overline{w}|^{n+2}}\, dt\, dw \\ &\lesssim \|u\|_{\mathcal{B}} \int_{\mathbf{H}} \int_0^1 \frac{1}{|z_t(a) - \overline{w}|^{n+2}}\, dt\, dw\end{aligned}$$

for $z \in \mathbf{H}$. It follows from Lemma 3.2 that

$$\sup_{z \in \mathbf{H}} z_n |\nabla D_j T_a u(z)| \lesssim a_n^{-1} ||u||_{\mathcal{B}}$$

for all j. Accordingly,

$$\sum \|D_j T_a u\|_{\mathcal{B}} \le C a_n^{-1} \|u\|_{\mathcal{B}}.$$

This completes the proof. □

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