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L^p AND H^p EXTENSIONS OF HOLOMORPHIC FUNCTIONS

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ABSTRACT. Given a subvariety M of a domain Ω, we will survey the extension problem of holomorphic functions from M to Ω. We briefly introduce some well-known results and then some new results. Finally, we make a question related to the extension problem.

1. AN EXTENSION PROBLEM

Let $\Omega \subset \mathbb{C}^n$ be a domain. Let M ba a subvariety of Ω and let $f : M \to \mathbb{C}$ be holomorphic. Does there exist an $F : \Omega \to \mathbb{C}$ such that F is holomorphic and $F|_M = f$? It turns out if Ω is pseuconvex, then we can always solve the problem. However, let us look at a critical example to see that for general domains we may not be able to solve the problem.

Example 1.1 ([10]). Let $\Omega = B(0,1)\backslash\bar{B}(0,1/2) \subset \mathbb{C}^2$. Let $M = \Omega\cap\{(z_1,z_2); z_2 = 0\} = \{(z_1,0); 1/2 < |z_1| < 1\}$. Let $f : M \to \mathbb{C}$ be given by $f(z_1,0) = 1/(z_1 - 1/2)$. Then f is holomorphic on M. Suppose that F is a holomorphic extension of f to all of Ω. By Hartog's phenomenon, F has a holomorphic continuation to $B(0,1)$. In particular, this F will be bounded in every neighborhood of the point $(1/2,0)$. Therefore, F cannot agree with f on M. □

2. AN L^2 EXTENSION THEOREM

In the case of the bounded pseudoconvex domain, Ohsawa-Takegoshi solved the L^2 extension problem.

Theorem 2.1 ([11]). *Let Ω be a bounded pseudoconvex domain and M be a complex linear subspace of codimension one of Ω and let ϕ be a plurisubharmonic function on Ω. Then for any holomorphic function f on M with*

$$\int_M |f|^2 e^{-\phi} dV_M < +\infty,$$

there exists a holomorphic extension F to all of Ω such that

$$\int_\Omega |F|^2 e^{-\phi} dV \leq C \int_M |f|^2 e^{-\phi} dV_M.$$

Here C depends on M and on the diameter of Ω.

We can give an example to show that in Theorem 2.1 the assumption of boundedness of the domain is necessary.

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Example 2.2. Let $\Omega = \{(z_1.z_2) \in \mathbb{C}^2; |z_2|^2 < 1/(1+|z_1|^2)\}$. Then Ω is an unbounded pseudoconvex domain with C^∞ boundary. We can see that

$$L^2(\Omega) \cap \mathcal{O}(\Omega) \subset \langle z_1^{\alpha_1} z_2^{\alpha_2}; \alpha_1, \alpha_2 = 0, 1, 2, \dots \text{ with } \alpha_1 < \alpha_2 \rangle.$$

Let $M = \{(z_1, z_2) \in \mathbb{C}^2; z_1 = 0\} \cap \Omega$. Choose $f \in \mathcal{O}(M) \cap L^2(M)$ with $f(0,0) \neq 0$. Suppose that there exists $F \in \mathcal{O}(\Omega) \cap L^2(\Omega)$ such that $F|_M = f$. Since $F(z_1, z_2) = \sum_{\alpha_1 < \alpha_2} C_{\alpha_1, \alpha_2} z_1^{\alpha_1} z_2^{\alpha_2}$, we have $F(0,0) = 0$. Therefore, F cannot agree with f on M. □

Furthermore, we can see that the estimate in Theorem 2.1 is best possible for weakly pseudoconvex domains in the L^2 sense.

Example 2.3. Let $\Omega = \{(z_1, z_2) \in \mathbb{C}^2; |z_1|^2 + 2e^{-1/|z_2|^2} < 1\}$. Let $M = \{(z_1, z_2) \in \mathbb{C}^2; z_2 = 0\} \cap \Omega$. For $F \in \mathcal{O}(\bar{\Omega})$, set $f(z_1, 0) = F(z_1, 0)$ for $(z_1, 0) \in M$. For any fixed $(z_1, 0) \in M$ the function $F(z_1, \cdot)$ is holomorphic in the disk of radius $r_{z_1} = \left(\log \frac{2}{1-|z_1|^2}\right)^{-1/2}$. By the mean value property, it follows that

$$\pi\ r_{z_1}^2\ f(z_1, 0) = \int_{|z_2| < r_{z_1}} F(z_1, z_2) dA(z_2).$$

Thus we have

(2.1)

$$\begin{aligned} \int_\Omega |F(z_1, z_2)|^p dV &= \int_{|z_1|<1} \int_{|z_2|<r_{z_1}} |F(z_1, z_2)|^p dA(z_2) dA(z_1) \\ &\geq \pi \int_{|z_1|<1} r_{z_1}^2\ |f(z_1, 0)|^p dA(z_1) \\ &= \pi \int_{|z_1|<1} \left(\log \frac{2}{1-|z_1|^2}\right)^{-1} |f(z_1, 0)|^p dA(z_1). \end{aligned}$$

Let f be some branch of $(1-z_1)^\alpha$ on M where $\alpha > 1$. Then $f \in L^2(M) \cap \mathcal{O}(M)$. Let $p > 2$. Choose α sufficiently close to 1 so that $2 < \alpha p$. It follows from (2.1) that f has no holomorphic extensions in $L^p(M) \cap \mathcal{O}(M)$ for $p > 2$. □

3. L^p AND H^p EXTENSIONS

For the strongly pseudoconvex case, we can solve the L^p and $H^p(0 < p \leq \infty)$ extension problems ([1], [2], [5], [6], [7], [8], [9]). Here $H^p(\Omega)$ means the Hardy space on Ω. Recently, we obtained $L^p(1 \leq p < \infty)$ and $H^p(1 < p \leq \infty)$ extensions of holomorphic functions from subvarieties of analytic polyhedra ([3]). The norm estimates are best possible in the case of analytic polyhedra.

Now let B be the unit ball in $\mathbb{C}^2$ and let $M = B \cap \{(z_1, z_2) \in \mathbb{C}^2; z_2 = 0\}$. Let $1 \leq p < \infty$. We consider a function defined by $f(z_1, 0) = 1/(1-z_1)^\alpha$ where $\alpha < 2/p$. Then $f \in \mathcal{O}(M) \cap L^p(M)$. Define $F(z_1, z_2) = f(z_1, 0)$ for $(z_1, z_2) \in B$. Then F is a holomorphic extension of f from the hyperplane M to the ambient domain B. Moreover, $F \in L^{3p/2}(B) \cap \mathcal{O}(B)$. This example illustrates that, at least as far as integrability is concerned, the holomorphic extension function is better behaved than the given holomorphic function in a hyperplane of the unit ball. We consider these phenomena in general cases of strongly pseudoconvex domains.

Let $\Omega \Subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^2 boundary. Let $\widetilde{M}$ be a submanifold of dimension m in a neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$ given as

$$\widetilde{M} = \{z \in \widetilde{\Omega};\ h_1(z) = \cdots = h_{n-m}(z) = 0\},$$

where $h_j \in \mathcal{O}(\widetilde{\Omega})$, and that $\partial h_1 \wedge \cdots \wedge \partial h_{n-m} \neq 0$ on $\widetilde{M}$. Set $M = \widetilde{M} \cap \Omega$ and $\partial M = \widetilde{M} \cap \partial\Omega$. We impose the transverse assumption that

$$\partial h_1 \wedge \cdots \wedge \partial h_{n-m} \wedge \partial\rho \neq 0 \quad \text{on} \quad \partial M. \tag{3.1}$$

Theorem 3.1. *Let $\Omega \Subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^2 boundary. Let M be a submanifold of Ω of dimension m. Assume that the transverse assumption (3.1) holds. Then for each $f \in \mathcal{O}(M) \cap L^p(M)$, $1 < p < \infty$ there exists $F \in \mathcal{O}(\Omega) \cap L^{(n+1)p/(m+1)}(\Omega)$ such that $F(z) = f(z)$ for $z \in M$ and $\|F\|_{L^{(n+1)p/(m+1)}(\Omega)} \lesssim \|f\|_{L^p(M)}$.*

We also have a corresponding result for Hardy spaces H^p.

Theorem 3.2. *Let $\Omega \Subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^2 boundary and (3.1) holds. Then for all $f \in H^p(M)$, $1 < p < \infty$, there exists $F \in H^{np/m}(\Omega)$ such that $F(z) = f(z)$ for $z \in M$ and $\|F\|_{H^{np/m}(\Omega)} \lesssim \|f\|_{H^p(M)}$.*

For explict extension formulas we make use of Berndtsson's interpolation integral formulas ([4]). In [5] we can see detailed proofs of Theorem 3.1 and Theorem 3.2. We see that norm estimates in Theorem 3.1 and Theorem 3.2 are best possible.

Example 3.3. Let B be the unit ball in $\mathbb{C}^2$ and let $M = \{(z_1, w) \in \mathbb{C}^2; z_2 = 0\} \cap B$. For $F \in \mathcal{O}(B)$, set $f(z_1, 0) = F(z_1, 0)$ for $(z_1, 0) \in M$. The function $F(z_1, \cdot)$ is holomorphic in a disc of radius $r_{z_1} = \sqrt{1 - |z_1|^2}$, so it follows from subharmonicity that

$$|f(z_1, 0)|^q \leq \frac{1}{2\pi r_{z_1}} \int_{|z_2| = r_{z_1}} |F(z_1, z_2)|^q ds(z_2),$$

where ds denotes the arc length measure. By Fubini's theorem,

$$\begin{aligned} \int_{\partial B} |F|^q d\sigma &= \int_{|z_1|<1} \int_{|z_2| = r_{z_1}} |F(z_1, z_2)|^q \frac{1}{r_{z_1}} ds(z_2) dA(z_1) \\ &\geq 2\pi \int_{|z_1|<1} |f(z_1, 0)|^q dA(z_1). \end{aligned} \tag{3.2}$$

Moreover, any arbitrary $F \in H^q(B)$ can be approximated by dilations which are holomorphic in a neighborhood of $\bar{B}$, so (3.2) remains valid for $F \in H^q(B)$.

Let $f(z_1, 0) = 1/(1 - z_1)^\alpha$ for $\alpha < 1/p$. Then $f \in H^p(M)$. Suppose that f has a holomorphic extension F in $H^q(B)$ for $q > 2p$. Choose α sufficiently close to $1/p$ so that $\alpha q > 2$. Then

$$\begin{aligned} \int_{|z_1|<1} |f(z_1, 0)|^q dA(z_1) &= \int_{|z_1|<1} \frac{dA(z_1)}{|1 - z_1|^{\alpha q}} \\ &\geq \int_{\substack{|z_1|<1 \\ |1 - z_1| < \epsilon}} \frac{dA(z_1)}{|1 - z_1|^{\alpha q}} \\ &\gtrsim \int_0^\epsilon dr \int_{\frac{3}{4}\pi}^{\frac{5}{4}\pi} \frac{d\theta}{r^{\alpha q - 1}}. \end{aligned} \tag{3.3}$$

¿From (3.2) and (3.3) we can see that f has no extension in $H^q(B)$ for $q > 2p$. Thus the H^p-extension in Theorem 3.2 is optimal. By similar considerations we can see that the L^p estimates of the extension function in Theorem 3.1 is also optimal. □

Problem 3.4. Let $\Omega_j = \{\rho_j < 0\}$ $(j = 1, \dots, N)$ be bounded strongly pseudoconvex domains in $\mathbb{C}^n$ with C^2 boundary. Let $\Omega = \Omega_1 \cap \dots \cap \Omega_N$. Then Ω is a strongly pseudoconvex domain with piecewise C^2 boundary. Let M be a submanifold of Ω defined by

$$M = \{z \in \bar{\Omega}; h_1(z) = \dots = h_{n-m}(z) = 0\},$$

where $h_j \in \mathcal{O}(\bar{\Omega})$. We assume the transverse condition

$$\partial h_1 \wedge \dots \wedge \partial h_{n-m} \wedge \partial \rho_{I_1} \wedge \dots \wedge \partial \rho_{I_k} \neq 0 \quad \text{on} \quad \bar{M} \cap \sigma_I,$$

where $\sigma_I = \{z \in \bar{\Omega}; \rho_j(z) = 0, j \in I\}$ for any multiindex I with $|I| = k$. Can we get some kinds of L^p and H^p extension theorems under these situations ?

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