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ESTIMATES OF INVARIANT METRICS ON PSEUDOCONVEX DOMAINS IN $\mathbb{C}^n$.

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Abstract. We introduce the Bergman, Caratheodory and Kobayashi metrics and get some quantities which bound from above and below these metrics in a small constant and large constant sense on various pseudoconvex domains in $\mathbb{C}^n$.

1. Introduction

In the last several decades, several complex variables has been greatly advanced and has influenced many areas of mathematics. Several comlex variables belongs to the area of analysis but the geometric nature of the field forces to study the geometric properties of the domains as Fefferman has employed in his one of the most important paper [10].

From the begining of the 1930's, the study of invariant metrics of Caratheodory, Bergman, Kobayashi's, which are the generalization of the Poincare's metric, has been done by several great mathematicians and has been one of the central field in several complex analysis. In this introductory paper we will introduce the concept of the Bergman, Caratheodory and Kobayashi metrics for a vector X at z, and will get some optimal estimates for these metrics. Throughout this paper, Ω will be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth defining function r, i.e., $\Omega = \{z \in \mathbb{C}^n; r(z) < 0\}$.

Let us first introduce the Poincare' metric on unit disc Δ in $\mathbb{C}^1$. Let X be a holomorphic vector field on Δ and $z \in \Delta$. Then the Poincare' metric $\|X(z)\|_P$ at z is given by

$$\|X(z)\|_P = \frac{|X(z)|}{1-|z|^2},$$

where $|X(z)|$ denotes the euclidean metric. For a given smooth curve $\gamma \subset \Delta$, we can calculate the length of the curve once we know the way to measure the length of the tangent vector at each point of the curve, i.e.,

$$L_{Euc}(\gamma) = \int_a^b |\gamma'(t)| dt$$

denotes the euclidean length of the curve, and

$$L_P(\gamma) = \int_a^b \|\gamma'(t)\|_{P,\gamma(t)} dt = \int_a^b \frac{|\gamma'(t)}{1-|\gamma(t)|^2} dt$$

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denotes the Poincare' length of the curve. Then, as usual, the Poincare' distance of $z, w \in \Delta$ is given by

$$d_P(z,w) = inf_\gamma L_P(\gamma).$$

For example, let $z = 0$, and $w = (r, 0)$ and let γ be the straight line connecting 0 and $(r, 0)$. Then the Poincare lenth of γ is given by

$$L_P(\gamma) = \int_0^r \frac{dt}{1-t^2} = tanh^{-1} r,$$

and this value approaches to infnity as r approaches to 1.

Now let $f : \Delta \longrightarrow \Delta$ be analytic. By virtue of the Schwartz lemma, it follows that

$$\frac{|f'(z)|}{1-|f(z)|^2} \le \frac{1}{1-|z|^2}, \quad \forall z \in \Delta$$

and this can be interpreted as

1. $\|df(z)X\|_{f(z)} \le \|X\|_z$, for all $z \in \Delta$,
2. $\|L_P(f \circ \gamma)\| \le L_P(\gamma)$, for any smooth curve γ in Δ,
3. $d_P(f(z), f(w)) \le d_P(z,w)$, $z, w \in \Delta$,
4. $f^*\|\cdot\| \le \|\cdot\|$, or $f^* d_P \le d_P$,
5. $\|\cdot\|$ is distance decreasing metric.

If f is biholomorphic then we can apply above properties and we conclude that the distance is invariant under the biholomorphic mapping.

We can extend these concepts in higher dimensions.

Definition 1.1. *Let Ω_1, Ω_2 be domains in $\mathbb{C}^n$ and let $f : \Omega_1 \longrightarrow \Omega_2$ be a biholomorphism. If*

$$F_{\Omega_2}(f(z); df(z)X) = F_{\Omega_1}(z; X), \;\; z \in \Omega_1,$$

where X is a holomorphic vector at z, then F is called invariant metric.

2. Invariant metrics in $\mathbb{C}^n$

Let X be a holomorphic tangent vector at a point z in Ω. Denote the set of holomorphic functions on Ω by $A(\Omega)$. Then the Bergman metric $B_\Omega(z;X)$, the Caratheodory metric $C_\Omega(z;X)$ and the Kobayashi metric $K_\Omega(z;X)$ are defined by

$$C_\Omega(z;X) = sup\{|Xf(z)| : f \in A(\Omega),\; \|f\|_{L^\infty(\Omega)} \le 1\}$$

$$K_\Omega(z;X) = inf\{1/r : \exists\, f : \Delta_r \subset \mathbb{C}^1 \to \mathbb{C}^n \text{ such that } f_*(\frac{\partial}{\partial z}|_0) = X\}$$

$$= inf\{\alpha > 0 : \exists\, f : \Delta \subset \mathbb{C}^1 \to \mathbb{C}^n \;\; f_*(\frac{\partial}{\partial z}|_0) = \alpha^{-1} X\}$$

$$B_\Omega(z : X) = b_\Omega(z;X)/(K_\Omega(z,\overline{z}))^{1/2},$$

where Δ_r denotes the disc of radius r in $\mathbb{C}^1$, and

$$K_\Omega(z,\overline{z}) = sup\{|f(z)|^2 : f \in A(\Omega), \;\; \|f\|_{L^2(\Omega)} \le 1\}$$

$$b_\Omega(z;X) = sup\{|Xf(z)| : f \in A(\Omega),\; f(z) = 0,\; \|f\|_{L^2(\Omega)} \le 1\}.$$

In the applications of these invariant metrics to function theories in several complex analysis, it is important to get a boundary behavior of them. In [11], K.T. Hahn got the following inequalities

$$C_\Omega(z;X) \le B_\Omega(z;X), \quad K_\Omega(z;X).$$

Therefore it is enough to get lower bounds of $C_\Omega(z;X)$ to get lower bounds of the other metric.

Theorem 2.1. *Let B be the unit ball in $\mathbb{C}^n$ and let $X = X_t + X_n$ be a holomorphic vector at $z \in B$, where X_t and X_n denote the tangential and normal components of X respectively. Then*

$$K_B(z;X) \approx |X_t| dist(z, bB)^{-\frac{1}{2}} + |X_n| dist(z, bB)^{-1}.$$

Proof. We first estimate for $z = 0$ and for $X = \frac{\partial}{\partial z_1}|_0$. Assume $f = (f_1, \dots, f_n) : \Delta \longrightarrow B$, and $f(0) = 0$, $f'(0) = \alpha^{-1}X$. Then the Cauchy's estimates shows that $|f_1'(0)| \leq 1$ and hence $K_B(0, \frac{\partial}{\partial z_1}|_0) \geq 1$. If we take the case that $f(\zeta) = (\zeta, 0, \dots, 0)$, then it follows that $K_B(0, \frac{\partial}{\partial z_1}|_0) = 1$. If X is a unit vector at 0, Then there is a unitary transformation U such that $K_B(0; \frac{\partial}{\partial z_1}|_0) = K_B(0; U_*(\frac{\partial}{\partial z_1}|_0)) = K_B(0; X)$. So $K_B(0; X) = 1$ because K_B is invariant.

Now we want to estimate at an arbitrary point. Let $\bar{a} = (a, 0, \dots, 0)$ and let F_a be a biholomorphism defined by

$$F_a(z) = \left(\frac{z_1 + a}{1 + az_1}, \frac{\sqrt{1 - |a|^2} z_2}{1 + az_1}, \dots, \frac{\sqrt{1 - |a|^2} z_n}{1 + az_1} \right).$$

If $Y = \sum_{j=1}^n b_j \frac{\partial}{\partial z_j}|_a$ is a holomorphic vector at $\bar{a}$, Then $Y = (F_a)_* X$, $X = \sum_{j=1}^n a_j \frac{\partial}{\partial z_j}|_0$, if and only if $a_1 = \frac{b_1}{1 - |a|^2}$ and $a_j = \frac{b_j}{\sqrt{1 - |a|^2}}$, $j \geq 2$. Hence $K_B(\bar{a}; Y) = K_B(0, X) = |X|$. Write $Y_n = b_1 \frac{\partial}{\partial z_1}|_a$ and $Y_t = \sum_{j=2}^n b_j \frac{\partial}{\partial z_j}|_a$. Then it follows that $K_B(\bar{a}; Y_n) = \frac{1}{1 - |a|^2} \approx d(\bar{a}; bB)^{-1}$ and $K_B(\bar{a}; Y_t) = \frac{1}{\sqrt{1 - |a|^2}} \approx d(\bar{a}; bB)^{-1/2}$.

If z is any point in B, we consider the unitary change of coordinates U_z which takes z to $(|z|, 0, \dots, 0)$. Then the invariance property of the Kobayashi metric shows that

$$K_B(z; X) \approx |X_n| \cdot d(z, bB)^{-1} + |X_t| \cdot d(z, bB)^{-\frac{1}{2}},$$

and this proves Theorem 2.1.

Theorem 2.2. *Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ with defining function r and let $b\Omega$ be strongly pseudoconvex at $z_0 \in b\Omega$. Then there is a neighborhood U of z_0 such that for all $z \in U \cap \Omega$,*

$$(*) \qquad |X_t||r(z)|^{-\frac{1}{2}} + |X_n||r(z)|^{-1} \approx C_\Omega(z; X), B_\Omega(z; X), K_\Omega(z; X).$$

Proof. We first approximate $b\Omega$ near z_0 in third orders by a boundary of a ball B which meets with $b\Omega$ at z_0. Then we can show that the invariant metrics of Ω near z_0 are approximated by the metrics of the ball in a neighborhood of z_0. If we use the method of the proof of Theorem 2.1, (*) will be followed. □

Recently, Several authors found some results about these metrics for some pseudoconvex domains in $\mathbb{C}^n$ [1,4,8,9,12,13]. In [2], Catlin got an optimal estimates of the invariant metrics in small constant and large constant sense and the author got a result for the estimates of these three metrics near a point of $z_0 \in b\Omega$ when the Levi form has corank one [5], and z_0 is of finite type m in the sense D'Angelo [7], and when the Levi-form of $b\Omega$ is comparable [6]. It uses a bumping theorem [3] and technical estimates of the $\bar{\partial}$-equation. Since their proof is very long and technical, we will omit detailed proof here and one can refer their results in references [2,3,4,5,6].

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