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THE COMMUTING PROBLEMS OF TOEPLITZ OPERATORS

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Abstract. We announce some recent works on commuting problems of Toeplitz operators on the Bergman or pluriharmonic Bergman space.

1. Introduction

The Bergman space A^2 is the Banach space consisting of all holomorphic functions in the usual Lebesgue space $L^2 = L^2(B, V)$ where the measure V is the normalized Lebesgue volume measure on the unit ball B in the n-dimensional complex space $\mathbb{C}^n$. The pluriharmonic Bergman space b^2 is the subspace of L^2 consisting of all complex-valued pluriharmonic functions in L^2. Recall that a function $u \in C^2(B)$ is pluriharmonic if its restriction to an arbitrary complex line that intersects the ball is harmonic as a function of single complex variable. As is well known, every pluriharmonic function on B can be expressed, uniquely up to an additive constant, as the sum of a holomorphic function and an antiholomorphic function. Moreover, one can check the relation $b^2 = A^2 + \overline{A^2}$. By an applications of the mean value property for holomorphic functions, it is not hard to see that the Bergman and pluriharmonic Bergman spaces are closed subspaces of L^2 and hence are Hilbert spaces with the usual inner product

$$< u, v >_2 = \int_B u\bar{v}\, dV,$$

respectively. See Chapter 3 and 4 of [10] for more information and related facts. We will write $P : L^2 \to A^2$ and $Q : L^2 \to b^2$ for the Hilbert space orthogonal projections respectively.

For $u \in L^\infty$, we define $T_u : A^2 \to A^2$ and $t_u : b^2 \to b^2$ respectively by

$$T_u f = P(uf) \qquad \text{and} \qquad t_u f = Q(uf).$$

These operators are called Toeplitz operators with symbol u. The Toeplitz operators are obviously bounded and in fact $||T_u|| \le ||u||_\infty$ in any cases.

Here, we are concerned with the characterizing problem of two symbols inducing commuting Toeplitz operators on the Bergman and pluriharmonic Bergman spaces, and announce some recent results and problems.

Question. Given two symbols, when do two Toeplitz operators commute?

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On the Hardy space of the unit disk, this problem has been completely solved with general symbols in [4]:

Theorem 1. *On the Hardy space of the unit disk, two Toeplitz operators with general bounded symbols commute if and only if either both symbols are analytic, or both symbols are conjugate analytic, or a nontrivial linear combination of symbols is constant.*

On the Bergman space, this problem was first initiated and solved for special cases in [3] on the unit disk. Later, this problem has been studied in each slightly different situations by several authors ([1], [2], [5], [7], [8], [11], [12]). Recently, this problem also has been studied on the pluriharmonic Bergman space and solved with holomorphic symbols in [6] and [9].

The purpose of this article is to announce some recent works and problems on this problem. In Section 2, we introduce two results on the Bergman space. In Section 3, we introduce a recent result on the pluriharmonic Bergman space. Also, we will introduce characterizations of normal Toeplitz operators and some open problems.

2. On the Bergman space

It is easy to see that each point evaluation is a bounded linear functional on A^2. Hence, there exists a unique function K_z– called the Bergman kernel– in A^2 which has the following reproducing property:

$$u(z) = < u, K_z >_{L^2} \qquad (u \in A^2). \tag{1}$$

It is well known that the Bergman kernel K_z has the following explicit formula

$$K_z(w) = \frac{1}{(1- < w, z >)^{n+1}} \qquad (w \in B) \tag{2}$$

where the notation $< z, w >$ denotes the Hermirian inner product in $\mathbb{C}^n$. See Section 3 of [10] for details. The formulas (1) and (2) lead us to the following integral representation of the projection P:

$$P\varphi(z) = \int_B \frac{\varphi(w)}{(1- < z, w >)^{n+1}}\, dV(w) \qquad (z \in B)$$

for functions $\varphi \in L^2$. So, the Toeplitz operators T_u has the following formula

$$T_u f(z) = \int_B \frac{u(w)f(w)}{(1- < z, w >)^{n+1}}\, dV(w) \qquad (z \in B)$$

for functions $f \in A^2$.

Since a Toeplitz operator with holomorphic symbol is a simple multiplication operator by the symbol, two Toeplitz operators with holomorphic symbols commute on A^2. Also, using the fact $T_u^* = T_{\bar{u}}$, one can easily see that two Toeplitz operators with antiholomorphic symbols also commute on A^2. So, we have a natural question on how the situation is with pluriharmonic symbols. The following theorem characterizes two pluriharmonic symbols for which the corresponding Toeplitz operators commute on A^2.

Theorem 2 ([11])**.** *Two Toeplitz operators with pluriharmonic symbols commute on A^2 if and only if either both symbols are holomorphic, or both symbols are antiholomorphic, or a nontrivial linear combination of symbols is constant.*

As an application of Theorem 1, we have a characterization of normal Toeplitz operators. See [5].

Corollary 3. *Let u be a pluriharmonic symbol. Then T_u is normal on A^2 if and only if $u(B)$ is a part of a line in $\mathbb{C}$.*

Recently, Axler, Čučković and Rho ([2]) first considered general symbols and prove the following characterization(in fact, they did all on any bounded open domain in the complex plane).

Theorem 4 ([2])**.** *Let $n = 1$. Suppose f be a nonconstant analytic symbol and g be a bounded symbol. Then, T_f and T_g commute on A^2 if and only if g is analytic.*

So, we have the natural question. See ([2]) for more related questions.

Question 5. *Let f be a nontrivial harmonic symbol and g be a bounded symbol such that T_f and T_g commute on A^2. Must g be of the form $af + b$ for some constants a, b?*

3. On the pluriharmonic Bergman space

Each point evaluation is also a bounded linear functional on b^2, there exists a unique function R_z in b^2 such that

$$u(z) = < u, R_z >_{L^2} \qquad (u \in b^2).$$

Since $b^2 = A^2 + \overline{A^2}$, there is a simple relation between R_z and Bergman kernel K_z: $R_z = K_z + \overline{K_z} - 1$. Thus, the explicit formula of R_z is given by

$$R_z(w) = \frac{1}{(1- < w, z >)^{n+1}} + \frac{1}{(1- < z, w >)^{n+1}} - 1 \qquad (w \in B).$$

So, we have the following integral representation for the projection Q:

$$Q\varphi(z) = \int_B \left(\frac{1}{(1- < w, w >)^{n+1}} + \frac{1}{(1- < w, z >^{n+1}} - 1 \right) \varphi(w) \, dV(w)$$

for $z \in B$ and $\varphi \in L^2$. The above representation shows that the projection Q can be also rewritten as

$$Q\varphi = P(\varphi) + \overline{P(\overline{\varphi})} - P(\varphi)(0) \tag{3}$$

for functions $\varphi \in L^2$.

In this section, we are concerned with the same problem on the pluriharmonic Bergman space and introduce a recent result in [9]. Situations of the commuting problem on the pluriharmonic Bergman space are quiet different from those on the Bergman space. For example, it is not hard to see that that two different holomorphic monomial symbols can not induce commuting Toeplitz operators.

We now introduce a characterization of two holomorphic symbols to induce commuting Toeplitz operators. The following two theorems were first proved in [6] on the unit disk and later in [9] on the ball.

Theorem 6 ([9])**.** *Let f, g be two nonconstant holomorphic symbols. Then t_f and t_g commute on b^2 if and only if $f = \lambda g + \mu$ for some constants λ, μ.*

We now state a characterization of normal Toeplitz operators on b^2 with pluriharmonic symbols.

Theorem 7. *Let u be a pluriharmonic symbol. Then t_u is normal on b^2 if and only if $u(B)$ is a part of a line in $\mathbb{C}$.*

Finally, we suggest a natural question.

Question 8. *Let f, g be harmonic symbols for which t_f and t_g commute on b^2. Then, must g be of the form $af + b$ for some constants a, b?*

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