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KOBAYASHI METRIC NEAR AN INFINITE TYPE BOUNDARY POINT

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ABSTRACT. We study the asymptotic behavior of the Kobayashi metric near exponentially-flat infinite type boundary points of bounded domains in the complex space of dimension 2. These depend upon the tangency of the streams of reference points to the boundary. This is a generalization of Graham's theorem on the asymptotic behavior of the Carathéodory and Kobayashi metrics of bounded strongly pseudoconvex domains in complex dimension n.

The primary goal of this article is in an investigation of boundary behavior of the Kobayashi-Royden metric near the boundary points of infinite type in bounded domains in $\mathbb{C}^2$.

Let G be a bounded domain in $\mathbb{C}^n$. The *Kobayashi metric* ([17]) on G is the function $F_K^G : G \times \mathbb{C}^n \to \mathbb{R}^+$, defined by

$$F_K^G(z;\xi) = \inf\{\alpha : \alpha > 0, \exists f \in H(D,G) \text{ with } f(0) = z, df|_0(\alpha) = \xi\}.$$

Note that the Kobayashi metric on G are in general Finsler but not Hermitian, i.e., they are a norm, which do not in general satisfy the parallelogram law.

We write some basic properties of the Kobayashi metric (see [9] and [13]). Let F^G be the Kobayashi metric on a bounded domain G in $\mathbb{C}^n$.

(a) Suppose that G is the unit disc in $\mathbb{C}$. Then the Kobayashi metric coincides with the Poincaré metric. The Kobayashi metrics of the unit ball B in $\mathbb{C}^n$ are given by

$$(F^B(z,\xi))^2 = \frac{\|\xi\|^2}{1-\|z\|^2} + \frac{|\langle z,\xi\rangle|^2}{(1-\|z\|^2)^2},$$

where $\langle z,\xi\rangle$ is the canonical hermitian inner product of z and ξ in $\mathbb{C}^n$.

(b) Let f be a holomorphic mapping from a bounded domain G in $\mathbb{C}^n$ into another bounded domain G' in $\mathbb{C}^m$. Then for $(z,\xi) \in G \times \mathbb{C}^n$,

$$F^{G'}(f(z);df|_z(\xi)) \le F^G(z;\xi).$$

In particular, if f is a biholomorphism, then

$$F^{G'}(f(z);df|_z(\xi)) = F^G(z;\xi).$$

(c) Let G and G' be bounded domains in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. Then

$$F^{G\times G'}((z,w);(\zeta,\xi)) = \max\{F^G(z;\zeta), F^{G'}(z;\xi)\}.$$

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Although the Kobayashi metric is a good tool in function theory, they are quite difficult to compute (for some examples of explicit computation, see [3]); so it becomes important to find asymptotic behavior of the metric. The first result in this line may be the following theorem due to I. Graham:

Theorem (Graham [9]). *Let G be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with C^2 boundary. Let $F(z;\xi)$ be either the Carathéodory or Kobayashi metric on G. Let $p \in \partial G$. Let ρ be a C^2 defining function for ∂G such that $\|\nabla_z \rho(p)\| = 1$. Then*

$$\lim_{z\to p} F(z;\xi)\, d(z,\partial G) = \frac{\|\xi_N(p)\|}{2}.$$

If $\xi_N(p) = 0$, i.e., ξ is a holomorphic tangent vector to ∂G at p, then

$$\lim_{z\to p; z\in\Lambda} (F(z;\xi))^2 d(z;\partial G) = \frac{L_{\phi,p}(\xi,\xi)}{2} = \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial z_j \partial \overline{z}_k}(p)\, \xi_j \overline{\xi}_k.$$

Here Λ denotes a cone of arbitrary aperture with vertex at p and axis the interior normal to ∂G at p.

This work tells us that the Kobayashi metric of a strongly pseudoconvex domain is asymptotically equivalent to the Poincaré-Bergman metric of the ball up to an appropriate normalization. This result was further analyzed later by several authors including G. Aladro [1], D. Ma [15], S. Fu [8], the author [14] and others using the scaling method of Pinchuk type (Kim [10], Kim-Yu [12], Boas-Straube-Yu [4], McNeal [16] to name only a few) which related the boundary behavior problem to an interior stability problem, initiated by Kim [10]. However most results obtained are for either strongly pseudoconvex case or the case of finite type boundary points (D. Catlin [5], S. Cho [7]).

Call a curve $q : (0,\epsilon) \to G$ a *stream* approaching $p \in \partial G$, if $q(t)$ satisfies $\lim_{t\to 0_+} q(t) = (0,0)$. With the same conditions to the Graham's theorem, we can write

$$\lim_{t\to 0} \frac{\left(\frac{\|\xi_{N,p(t)}\|}{2d(q(t),\partial G)}\right)^2 + \frac{L_{\partial G,p(t)}(\xi_{T,p(t)},\xi_{T,p(t)})}{d(q(t),\partial G)}}{F(q(t);\xi)^2} = 1, \tag{1}$$

where $d(q(t),\partial G)$ is the distance from $q(t)$ to ∂G, and $p(t)$ is the closest boundary point to $q(t)$ (see [14]). The formula (1) explains the asymptotic behavior of the Kobayashi metric for *every* streams and *every* vectors; for the case of a tangential stream with $\xi_{N,p} = 0$, which is not considered in the Graham's theorem, it is possible that $\xi_{N,p(t)} \neq 0$ so that we may have

$$\lim_{t\to 0} \frac{\|\xi_{N,p(t)}\|}{2d(q(t),\partial G)} \neq 0.$$

The class of domains we consider in this paper possess exponentially-flat infinite type boundary points (see [11]). A boundary point, say $(0,0)$ of a bounded domain $G \subset \mathbb{C}^2$ is said to be an *exponentially-flat boundary point* of G *of order* m if there is a C^∞ function $\phi : \mathbb{R} \to \mathbb{R}$ satisfying

(A) $\phi(x) = 0$, $\forall x \leq 0$,
(B) $\phi''(x) > 0$, $\forall x > 0$, and
(C) for $x > 0$, the function $\psi(x) = -1/\log\phi(x)$ (defined to be 0 for $x = 0$) extends to a function that is C^∞ smooth at 0, vanishing to a finite order m at that point,

such that for some neighborhood U of the origin $(0,0)$,

$$G \cap U = \{(z_1, z_2) \in U \,:\, \operatorname{Re} z_1 + \phi(|z_2|^2) < 0\}.$$

The case of $\phi(x) = \exp(-\frac{1}{x})$, $x > 0$, is the prototype example.

This concept, while not equivalent to the concept of general infinite type, surely is a typical example of a non-Levi flat infinite type boundary point at least in complex dimension two. Near such a boundary point, almost all boundary points are strongly pseudoconvex, but there are also infinite type points.

In the case of infinite type bounded domain, the asymptotic behavior of the Kobayashi metric depends upon the tangency of the stream. We consider various all possible degrees of the tangency of the streams $q(t)$.

Let $q(t) = (q(t)_1, q(t)_2) \in G$. Note that $\lambda(t) := |q_2(t)|$ represent the distance from the reference point $(q_1(t), q_2(t))$ to the center line $\{(0,0) + s\nabla\rho(0,0) : s \in \mathbb{R}\}$. Let $d^*(t)$ be the distance from the reference point to the boundary of the slice of G, given by $G \cap [q(t) + T^{\mathbb{C}}_{p(t)}(\partial G)]$, where $p(t)$ is the closest boundary point to the reference point $q(t)$. Therefore the ratio $d^*(t) : \lambda(t)$ determines the tangency of the stream $q(t)$ with respect to the boundary ∂G: we can say that the degree of tangency of the stream increases, as $d^*(t)/\lambda(t)$ approaches 0.

Definition 1. If $d^*(t)/\lambda(t)$ approaches infinity as t tends to 0, then the stream $q(t)$ is said to be *central*. If $d^*(t)/\lambda(t)$ approaches $\mathfrak{a}$ $(0 < \mathfrak{a} < \infty)$, then the stream $q(t)$ is said to be *intermediate*. And if $d^*(t)/\lambda(t)$ approaches 0, then the stream $q(t)$ is said to be *extreme-tangential*.

Central streams consist of non-tangential streams, finite-type tangential streams, exponentially-flat tangential stream of order less than m (m is the vanishing order of $\psi(x) = -1/\log\phi(x)$). Intermediate streams have the same tangency as the exponentially-flat of order m to the domain at $(0,0)$. The extreme-tangential streams have even more tangency to the domain G at $(0,0)$.

In the following we state our main result. Let G be a bounded domain in $\mathbb{C}^2$ with an exponentially-flat boundary point $p \in \partial G$ of order m. Let $q(t) = (q_1(t), q_2(t))$ be an C^∞ stream in G, approaching p. Let $\mathcal{N}(t) = -\frac{q(t)-p(t)}{\|q(t)-p(t)\|}$ be the outward unit normal vector to ∂G at $p(t)$. Let $\mathcal{T}(t) = (T_1(t), T_2(t)) \in T^{\mathbb{C}}_{p(t)}(\partial G)$ be the unit tangent vector to ∂G at $p(t)$ with $\arg T_2(t) = \arg q_2(t)$. Then $\{\mathcal{N}(t), \mathcal{T}(t)\}$ forms an orthonormal frame for $T^{\mathbb{C}}_{p(t)}(\mathbb{C}^2) = \mathbb{C}^2$. For any vector $\xi \in \mathbb{C}^2$, we let

$$\xi_{N,p(t)} = \langle \xi, \mathcal{N}(t)\rangle, \quad \xi_{T,p(t)} = \langle \xi, \mathcal{T}(t)\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the standard Hermitian inner product in $\mathbb{C}^2$. Let $d(q(t), \partial G)$ and $d^*(q(t), \partial G)$ be the distances from the reference point $q(t)$ to the boundary ∂G of G and to the boundary of the slice of G, given by $G \cap \{q(t) + T^{\mathbb{C}}_{p(t)}(\partial G)\}$, respectively. We may assume that for each t, there is unique boundary point $p(t) \in \partial G$ such that $d(q(t), \partial G) = d(q(t), p(t))$. Let $\lambda(t)$ be the distance from $q(t)$ to the center line, given by $\{p + sN_p | s \in \mathbb{R}\}$, where N_p is the the outward unit normal vector to ∂G at $p \in \partial G$.

Theorem 1. *Let $F(z;\xi)$ be the Kobayashi metric on G.*

If the stream $q(t)$ is "central", then

$$F(q(t);\xi) \sim \max\left\{\frac{|\xi_{N,p(t)}|}{2d(q(t),\partial G)}, \frac{|\xi_{T,p(t)}|}{d^*(q(t),\partial G)}\right\}.$$

If $q(t)$ is an "intermediate" stream, then

$$F(q(t);\xi) \sim \max\left\{\frac{|\xi_{N,p(t)}|}{2d(q(t),\partial G)}, \frac{1+1/\mathfrak{a}}{1+2/\mathfrak{a}}\cdot\frac{|\xi_{T,p(t)}|}{d^*(q(t),\partial G)}\right\}.$$

where $\mathfrak{a} = \lim_{t\to 0} d^*(q(t),\partial G)/\lambda(t)$.

Suppose $q(t)$ is an "extreme-tangential" stream of G.

(a) *If $d^*(t)/(\lambda(t))^{1+2m}$ diverges to infinity, then*

$$F(q(t);\xi) \sim \max\left\{\frac{|\xi_{N,p(t)}|}{2d(q(t),\partial G)}, \frac{|\xi_{T,p(t)}|}{2d^*(q(t),\partial G)}\right\}.$$

(b) *If $d^*(t)/(\lambda(t))^{1+2m}$ converges to $\mathfrak{a}$, $0<\mathfrak{a}<\infty$, then*

$$F(q(t);\xi) \sim F_K\left((-1,0);\left(\frac{|\xi_{N,p(t)}|}{d(q(t),\partial G)}, \frac{|\xi_{T,p(t)}|}{d^*(q(t),\partial G)}\right)\right),$$

where F_K is the Kobayashi metric for the domain

$$\{(u_1,u_2)\in\mathbb{C}^2 \,:\, \operatorname{Re} u_1 + A(\exp\{\mathfrak{c}\operatorname{Re} u_2\} - 1 - \mathfrak{c}\operatorname{Re} u_2) < 0\},$$

for some $0<\mathfrak{c}<\infty$ and $A = \frac{1}{\mathfrak{c}^2/(2!)+\mathfrak{c}^3/(3!)+\cdots}$.

(c) *If $d^*(t)/(\lambda(t))^{1+2m}$ converges to 0, then*

$$F(q(t);\xi) \sim \sqrt{\left(\frac{|\xi_{N,p(t)}|}{2d(q(t),\partial G)}\right)^2 + \left(\frac{|\xi_{T,p(t)}|}{\sqrt{2}d^*(q(t),\partial G)}\right)^2}.$$

As one would expect from the results preceding this work, the analysis of limiting behavior depends upon the tangency of the stream of reference points, which appears even in the finite type cases. (See [2],[3], [6].) The asymptotic behaviors changes from that of bidisc ($\mathbb{H}\times D$) to that of the unit ball up to an appropriate normalization as the tangency of the streams grows.

We can ask the asymptotic behavior of Kobayashi distance for the same case. Theorem 1 also gives us a motivation to study the domains, defined by

$$\{(u_1,u_2)\in\mathbb{C}^2 \,:\, \operatorname{Re} u_1 + (\exp\{\operatorname{Re} u_2\} - 1 - \operatorname{Re} u_2) < 0\}.$$

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