Trends in Mathematics
Information Center for Mathematical Sciences
Volume 4, Number 1, June 2001, Pages 38–43

A SURVEY ON THE MATRIX COMPLETION PROBLEM

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Abstract. Completion problems arise in a variety of applications, such as statistics (e.g. entropy methods for missing data), chemistry (the molecular conformation problem), systems theory, discrete optimization (relaxation methods), data compression, etc., as well as in operator theory and within matrix theory (e.g. determinantal inequalities). In addition to applications, completion problems have provided an excellent mechanism for understanding matrix structure more deeply. In this article, we survey the recent works on matrix completion problems.

1. INTRODUCTION

A *partial matrix* is a rectangular array in which some entries are specified, while the remaining unspecified entries are free to be chosen from an indicated set (such as the real numbers or complex numbers). A *completion* of a partial matrix is a specific choice of values for the unspecified entries resulting in a conventional matrix. Problems based on a variety of properties have been studied. But the positive definite completion problem has received the most attention, due, in part, to its role in several applications in probability and statistics, image enhancement, systems engineering, geophysics, etc. and to its relation with other completion problems including spectral norm contractions and Euclidean distance matrices which is important for the molecular conformation problem in chemistry. Another view of the positive definite completion problem is as a mechanism for addressing a fundamental problem in Euclidean geometry: which potential geometric configurations of vectors (i.e. configurations with angles between some vectors specified) are realizable in a Euclidean space?

In a typical matrix completion problem, description of circumstances is sought in which choices for the unspecified entries may be made from the same set S so that the resulting ordinary matrix over S is of a desired type. In the vast majority of cases that have been of interest in matrix completion problem.

A *matrix completion problem* asks whether a given partial matrix has a completion of a desired type; for example, the positive definite completion problem asks which partial Hermitian matrices have a positive definite completion. The properties of matrix completion problems have been inherited permutation similarity, diagonal matrix multiplication and principal submatrices.

Completion problems have proved to be a useful perspective to study fundamental matrix structure. The positions of the specified entries in a partial matrix

2000 *Mathematics Subject Classification.* 15A04, 15A21, 15A30.
Key words and phrases. matrix completion, partial matrix, positive definite.
Received August 30, 2000

are naturally described by a graph, and there are strong relationships between the structure of this graph and conditions for completability. Chordal graphs were initially important in matrix analysis in connection with Gaussian elimination on sparse matrices, and they continue to play a fundamental role in sparse matrix computations.

Consider a property Q that a conventional matrix may enjoy (e.g., positive definite, Euclidean distance, contraction, totally positive, bounded rank, etc). In the vast majority of cases that have been of interest in matrix completion theory, Q has been a property that is inherited by submatrices. For example, Q is the positive definiteness of square matrices is inherited by principal submatrices and Q is the spectral norm contraction is inherited by all submatrices. In case the property is inherited, we say that a partial matrix is a "*partial Q-matrix*" if every specified (principal or general, depending upon your Q) submatrix has the property Q, and if a partial matrix A is to have a completion with the property Q, it is an obvious necessary condition that the partial matrix A have the property Q. In recent history, matrix completion problems have dealt with partial matrices with a given part that is triangular or banded (a natural choice when one considers Hermitian matrices).

We may classify the matrix completion problems in three categories as following;

A. Hermitian problems:

B. Rank problems:

C. Eigenvalue and singular value completion problems

2. HERMITIAN PROBLEMS

A-1. Positive definite completion problem

First we summarize results on the positive definite completion problems.

Definition. *A graph is called chordal if it contains no minimal cycles of length 4 or more.*

Theorem[14, GJW, 1984]. *Every partial positive definite matrix with graph G has a positive definite completion if and only if G is chordal.*

We now consider the question of whether a partial positive definite matrix with nonchordal graph has a positive definite completion. The well known simple nonchordal graph is

$$C_n = (N, E), \quad E = \{\{1,2\},\{2,3\},\dots,\{n-1,n\},\{n,1\}\}.$$

Let

$$A(C_n) = \begin{pmatrix} 1 & cos\theta_1 & ? & ? & ? & cos\theta_n \\ cos\theta_1 & 1 & cos\theta_2 & ? & ? & ? \\ ? & cos\theta_2 & 1 & \ddots & ? & ? \\ ? & ? & \ddots & \ddots & \ddots & ? \\ ? & ? & ? & \ddots & \ddots & cos\theta_{n-1} \\ cos\theta_n & ? & ? & ? & cos\theta_{n-1} & 1 \end{pmatrix}$$

be the matrix with graph C_n where $0 \leq \theta_i \leq \pi$. In 1993, W. Barrett, C. R. Johnson and P. Tarazaga proved the following.

Theorem[4, BJT, 1993]. *Let $n > 3$. The real symmetric partial positive definite $n \times n$ matrix A given by above has a positive definite completion if and only if*

$$\Sigma_{i=1}^{k} < (k-1)\pi + \Sigma_{i=k+1}^{n}\theta_i$$

for $k \in \{1, 2, \cdots, n\}$ and k is odd, where $0 \leq \theta_n \leq \theta_{n-1} \leq \cdots \leq \theta_2 \leq \theta_1 \leq \pi$.

A-2. M-matrix completion problem

Definition. *Let A be an $n \times n$ real matrix. We call that A is an M-matrix if A has nonpositive off-diagonal entries and an inverse all of whose entries are nonnegative. Also A is called an inverse M-matrix if A^{-1} is an M-matrix.*

The M-matrix completion problem is to determine which partial matrices have an M-matrix completion and inverse M-matrix completion. A natural assumption for the partial matrix is all diagonal entries are 1. We list some results on M-matrices, inverse M-matrices and P-matrices ,using graph theory. Leslie Hogben characterized the M-matrix completion in 1998 [12, Ho]. He also gave the following theorem on inverse M-matrix patterns.

Theorem[12, Hogben, 1998]. *Let Q be a pattern that includes all diagonal entries and let G be its digraph. The following are equivalent:*

(1) the pattern Q has M-completion;
(2) the pattern Q is permutation similar to a block triangular pattern with all the diagonal blocks completely specified;
(3) any strongly connected subdigraph of G is complete.

A pattern with some diagonal entries unspecified has M-completion if and only if the principal subpattern defined by the specified diagonal positions has M-completion.

Theorem[12, Hogben, 1998]. *Let G be an undirected graph on n vertices. Every $n \times n$ positionally symmetric partial inverse M-matrix the graph of whose entries G has an inverse M-matrix completion if and only if G is block clique. Moreover, there is a unique completion which has a zero entry in the inverse in every position in which the partial inverse M-matrix has an unspecify entry; this completion is an inverse M-matrix completion with maximum determinant.*

He also improved the above theorem in [11, Ho].

Let $N = \{1, ..., n\}$. Which subsets Q of $N \times N$ have the property that whenever a_{ij} for (i, j) in Q form a "partial inverse M-matrix" (i.e., $a_{ij} > 0$ and if $L \times L$ is a subset of Q then the inverse of $\{a_{ij} : i, j \in L\}$ is an M-matrix), a_{ij} can be defined for the (i, j) not in Q so that the inverse of $A = [a_{ij}]$ is an M-matrix? In terms of a digraph G with vertex set N and arc set Q, they have the following results say

(1) it is necessary that the induced subdigraph of any alternate path to an arc be complete,

(2) a cycle need not be contained in a complete subdigraph if (i, i) is not in Q for at least one of its vertices i (by contrast, if (i, i) is in Q for all vertices i, it is necessary that the induced subdigraph of a cycle be complete),

(3) when $n = 4$, it is necessary and sufficient that condition (1) is true and every simple cycle has a vertex i with (i, i) not in Q, and

(4) the general question can be reduced to the case in which G is strongly connected, much as it has been when Q contains (i, i) for every i.

A-3. The combinatorially symmetric P-matrix completion problem

An $n \times n$ real matrix A is called a *P-matrix* if all of its principal minors are positive. This class generalizes many other important classes of matrices (such as positive definite, M-matrices, and totally positive), has useful structure (such as inverse closure, inheritance by principal submatrices, and wedge type eigenvalue restrictions), and arises in applications (such as the linear complementary problem, and issues of local invertibility of functions). Since the property of being a P-matrix is inherited by principal submatrices, it is necessary that the partial matrix be a partial P-matrix, i.e., every fully specified principal submatrix must itself be a P-matrix. Of all these assumptions, the only one that is truly restrictive is the combinatorial symmetry which means that a_{ji} is specified if the entry a_{ij} is specified.

Let A be an $n \times n$ real partial P-matrix with one pair of symmetrically placed unspecified entries.

Theorem [15, JK, 1996]. *Every combinatorially symmetric partial P-matrix has a P-matrix completion.*

The combinatorially symmetry assumption is relaxed, the conclusion no longer holds, and the question of which directed graphs for the specified entries ensure that a partial P-matrix has a P-matrix completion is, in general, still open. While C. R. Johnson was visiting Sungkyunkwan University in the spring of 1998, Seol and Lee posed the problem and gave a partial solution of the problem. In 1998, a better partial answer on sign symmetric P-matrix completion problem were given by Jordan, Torregrosa and Urbano at the International Linear Algebra Society conference, Madison. In 2000, Shaun M. Fallat, Charles R. Johnson, Juan R. Torregrosa and Ana M. Urbano [7, FJTU, 2000] gave the answer for the P-matrix completions under weak symmetry assumption which has the graph of specified entries is an n-cycle. They also proved that a combinatorially weakly symmetric partial P-matrix(Π-matrix) has a P-matrix completion if the graph of its specified entries is a 1-chordal graph.

Theorem [7, FJTU, 2000]. *Let G be an undirected 1-chordal graph. Then any partial Π-matrix, the graph of whose specified entries is G, has a Π-matrix completion.*

3. RANK PROBLEMS

Most of rank completion problems were done by Weordeman and his coworkers. We list some reference of them; [5, CJRW, 1989], [17, W, 1989], [19, W, 1993], [18, W, 1994]. And recently James F. Geelen described an algorithm that find a maximum rank matrix by perturbing an arbitrary completion, in the 1998 Research Report of Dept. of Combinatorics and Optimization at University of Waterloo [Report number CORR98-(7)].

4. EIGENVALUE AND SINGULAR VALUE COMPLETION PROBLEMS

Many works on this type of problems have been done with eigenvalues. Because the structured eigenvector results are used to give a new proof of the maximum minimum eigenvalue completion problem for partial Hermitian matrices a chordal graph. Some of the interesting results can be found in the followings; [2], [3], [12], [14], [18] and [19].

5. CONCLUSION

In the matrix completion problems, many results have been given and many researcher have given their attention on it. Fortunately, using elementary ideas of real algebraic geometry, one may demonstrate the existence of finitely many polynomial inequality conditions (depending upon G) for all properties Q thus far of interest in [15]. However, obtaining a practical list of conditions is much more difficult. (Of course, the answer to the initial question discussed above is a special case.) For a few properties (e.g. positive definiteness) there are now numerical (or computational) approaches based upon matrix theory and optimization (e.g. semidefinite programming. These have been and are being developed and tested by the William and Mary matrix theory research group and others and are useful for several applications. However, our long term interest is an analytical solution and a general approach to many different properties. In additional to its value in the applications this will also likely contribute much greater insight into matrix structures.

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