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# DECAYS OF INCOMPRESSIBLE FLUIDS

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Abstract. We study the decay rate for incompressible fluids on various domains. By considering the heat equation we estimate the decay for the Stokes equations. Since the Stokes equations are the linear version of the Navier-Stokes equations, by comparing those two equations we can estimated the decay for the Navier-Stokes equations.

## 1. Introduction

We survey the asymptotic behavior of weak solutions of the Navier–Stokes equations, and of the Stokes equations. The Navier–Stokes equations(NSE) consists of the momentum equation:

$$u_t - \Delta u + (u \cdot \nabla)u + \nabla p = 0, \qquad \text{in } \Omega \times (0,\infty), \tag{1.1}$$

and the continuity condition, with initial and boundary data:

$$\begin{aligned} \nabla \cdot u &= 0, && \text{in } \Omega \times (0,\infty), \\ u(x,0) &= u_0, && \text{for } x \in \Omega, \\ u(x,t) &= 0, && \text{for } (x,t) \in \partial\Omega \times (0,\infty), \end{aligned} \tag{1.2}$$

and the Stokes equations(SE) consists of the momentum equation:

$$u_t - \Delta u + \nabla p = 0, \qquad \text{in } \Omega \times (0,\infty), \tag{1.3}$$

and (1.2), where $n \geq 2$, and $\Omega \subset \mathbb{R}$ is an open subset of $\mathbb{R}^n$. Here, $u \overset{def}{=} (u_1, \dots, u_n)$ and $p$ denote the velocity and pressure, respectively, while $u_0$ is a given initial velocity satisfying the continuity condition. We denote by (NSE) the Navier–Stokes equations (1.1) and (1.2), by (SE) the Stokes equations (1.3) and (1.2).

The decay problem for weak solutions of the Navier–Stokes equations was first proposed by Leray [13] for the Cauchy problem in $\mathbb{R}^3$. If the domain $\Omega$ itself or one direction of the domain is bounded, then the decay rate of solutions is exponential. It is obtained easily with the help of Poincare's inequality. For a domain which is not bounded in any direction, it has a different story. The typical domains are the whole space $\mathbb{R}^n$, the half space $\mathbb{R}^n_+$, and the exterior domains of which complements are compact. Masuda [14] gave the explicit estimate for the decay of solutions in exterior domains for the first time. Schonbek [15, 16, 17, 18] worked the decay problem in $\mathbb{R}^n$. She obtained the lower and the upper bounds. Kajikiya and Miyakawa [10], and Wiegner [20] discussed the same problem. However, all

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of the above results except Kajikiya and Miyakawa [10] rely on the theory of the Fourier transform. In the case of the half space the Fourier transform method does not work well.

Borchers and Miyakawa [6] studied the decay problem in half spaces $\mathbb{R}^n_+$ using the semigroup theory. Bae and Choe [2] also showed the decay rate if the solution lies in an appropriate weighted space.

In general, to study the decays for the Navier–Stokes equations, the estimates for the solutions of the Stokes equations are essential. If the spatial domain is $\mathbb{R}^n$, then solution $u$ of (SE) is reduced to that of the heat equation with initial data $u_0$. Then for all $1 \le r \le \infty$,

$$\|\nabla u(t)\|_r \le Ct^{-1/2}\|u_0\|_r \qquad \text{for } t > 0 \tag{1.4}$$

holds. For $1 < r < \infty$, (1.4) is valid for half spaces. (See Ukai [19]). For $1 < r < n$, (1.4) is also valid for exterior domains. (See Borchers and Miyakawa [6].) Schonbek [17, 18] also considered the decay rates if the average is zero, $\int_{\mathbb{R}^n} |u_0|^2|x|dx < \infty$, and under some restrictions on $u_0$. In Bae and Choe [2], we estimated the decay rate if the solution lies in an appropriate weighted space; for $1 < r \le q < \infty$,

$$\|u(t)\|_{L^q(\mathbb{R}^n_+)} \le Ct^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{2}}\left(\int_{\mathbb{R}^n_+} |y_n|^r|u_0(y)|^r\,dy\right)^{1/r}.$$

Giga, Matsui and Shimizu [8] first showed (1.4) for $r = 1$ in the half spaces $\mathbb{R}^n_+$. Since the projection associated with the Helmholtz decomposition is not bounded in $L^1$ type spaces if the domain has nonempty boundaries, it is not easy to obtain such result for $r = 1$ and $r = \infty$. In [8], they estimated the rates in the Hardy space, and then as a corollary, they obtained (1.4) for $r = 1$. In [1], we also obtain estimates for $r = 1$ in the half spaces. However, the the time decay rate is not $t^{-1/2}$ but $t^{-1}$.

## 2. On $\mathbb{R}^n$

In the whole space $\mathbb{R}^n$, the Stokes equations are reduce to the heat equations. Since the heat equations are well understood, by comparing (1.1) and the heat equations and by using the Fourier transform, the decay rates of solutions of (NSE) are obtained. If $A$ denotes $-\Delta$, then it is obtained

$$\|e^{-At}u_0\|_{L^q} \le Ct^{-(n/r-n/q)/2}\|u_0\|_{L^r} \qquad \text{for } 1 \le r \le q \le \infty$$

with the help of the well–known fundamental solution of the heat equation.

Kato [11] showed if $u_0$ is small in $L^n$-norm, then there is a unique strong $L^n$ solution $u$ such that

$$\|u(\cdot,t)\|_{L^q} \le Ct^{-(n/r-n/q)/2}\|u_0\|_{L^r} \qquad \text{for } q \le r \le \infty.$$

Wiegner [20] had that for $u_0 \in L^2 \cap L^1$,

$$\|u(\cdot,t)\|_{L^2} \le Ct^{-n/4}.$$

Schonbek in [15], [16], [17], and [18], studies the decays for (1.1) on the whole space $\mathbb{R}^n$. In [15], she showed that there exists a Leray-Hopf solutions of (1.1) with arbitrary data in $L^1 \cap L^2$ which satisfies

$$\|u(\cdot,t)\|_{L^2(\mathbb{R}^3)} \le Ct^{-1/4}.$$

In [16], she improved the result by showing that the rate of decay for solutions with large data in $L^r \cap L^2$, $1 \le r < 2$, is the same as for solutions of the heat equation,

$$\|u(\cdot,t)\|_{L^2(\mathbb{R}^3)} \le Ct^{-3/2(1/r-1/2)}.$$

She also obtained in [17] and [18] that for non-zero average initial data there exist constants $C_1, C_2$ depending on the initial data such that

$$C_1(t+1)^{-3/4} \le \|u(\cdot,t)\|_{L^2(\mathbb{R}^3)} \le C_2(t+1)^{-3/4},$$

and that if $u(x,t)$ is a weak solution in the sense of Cafarelli-Kohn-Nirenberg with zero average initial data outside a class of functions of radially equidistributed energy, then there exist constants $C_3, C_4$ depending on the initial data such that

$$C_3(t+1)^{-5/2} \le \|u(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} \le C_4(t+1)^{-5/2}.$$

The algebraic lower bound is a consequence of the nonlinear structure of the equations. In contrast, solutions to the heat equation decay at an exponential rate if the initial data is highly oscillatory. The inertial term $\operatorname{div}(u \otimes u)$ in (1.1) appears to convert short waves into long waves, reducing the decay rate. Her approach in the case of zero average data is first to find conditions for the data such that the corresponding solution to the heat equation decays at a very slow rate. These conditions will be met by the solution $u(x,t)$ of (NSE) at some time $t_0 \ge 0$. That is, short waves are transformed into long waves. Hence the solution to the heat equation, which takes on as initial data $u(x,t_0)$ for some appropriate $t_0 \ge 0$, has lower bound on their rate of decay.

For initial data which have radially equidistributed energy, an extension of an example suggested by Majda for the 2-D Navier–Stokes shows that solutions can be constructed which decay exponentially, showing that the condition of radially equidistributed energy on the data is necessary.

Schonbek had used the Fourier transform. Kajikiya and Miyakawa [10] obtained $\|u(t)\|_{L^2} \le Ct^{-(n/r-n/2)/2}$ if $u_0 \in L^2 \cap L^1$ for $1 \le r < 2$ using the spectral theory of self-adjoint operators in Hilbert space.

Chen [7] and Beirão da Veiga [3] also obtained some results.

## 3. On Exterior Domains

Masuda [14] gave the decays in exterior domain with $t^{-1/8}$. Heywood [9] obtained the rate $t^{-1/2}$.

Borchers and Miyakawa in [4] and [5] showed that there is a weak solution $u$ such that

(i) $\|u(t)\|_{L^2} \to 0$,
(ii) if $\|e^{-tA}u_0\|_{L^2} = O(t^{\alpha})$ for some $\alpha > 0$, then

$$\|u(t)\|_{L^2} = \begin{cases} O(t^{-\alpha}) & \text{if } \alpha < n/4, \\ O(t^{-n/4}) & \text{if } \alpha \ge n/4, \end{cases},$$

where $A$ is the Stokes operator. They used the Fourier analysis for closed linear operators in Banach space and extended those of Schonbek [15, 16, 17, 18], and Kajikiya and Miyakawa [10], and Borchers and Miyakawa [4].

## 4. On the Half Space $\mathbb{R}^n_+$

All of the results in Section 2, except Kajikiya and Miyakawa [10] rely on the theory of the Fourier transform. In the case of the half space the Fourier transform method does not work well.

Borchers and Miyakawa [6] studied the decay problem in half spaces $\mathbb{R}^n_+$. They obtained that if $u_0 \in L^2 \cap L^r$, then

$$\|u(t)\|_2 \le C_2(1+t)^{-\frac{n}{2}(1/r-1/2)}$$

provided $1 \le r < 2$. They showed the boundedness and analyticity of the semigroup, and studied the fractional powers of the Stokes operator. For their estimation, they used the Ukai's formula in [19]. However, for $r = 2$, they obtained that $\|u(t)\|_2 \to 0$ as $t \to \infty$, but they could not estimate its rate. Kozono [12] also obtained similar decays for small initial data.

Bae and Choe [2] showed that the decay rate of $L^2$-norm of the solutions for (1.1) in the half space. We used the heat kernel, Ukai's solution formula for the Stokes equations ([19]), and the semigroup theory.

**Theorem 4.1.** *Suppose that $u_0 \in L^2 \cap L^r$ for some $r$ with $1 < r \le 2$, and $\nabla \cdot u_0 = 0$, and that*

$$\int_{\mathbb{R}^n_+} |y_n u_0(y)|^r dy < \infty.$$

*Then, for $n \ge 2$ we have that for $t > 0$,*

$$\|u(t)\|_2 \le C t^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{2})-\frac{1}{2}}.$$

We also estimate the decay rate of difference of the Stokes flow and the Navier-Stokes flows. This result improves the lower bound estimates of the decay rate in [6].

**Theorem 4.2.** *With the same assumptions in Theorem 4.1, we have that for $t > 0$,*

$$\|u(t) - v(t)\|_2 \le C t^{-\frac{n}{4}-\frac{1}{2}} = o(t^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{2})-\frac{1}{2}}), \qquad \text{for } 1 < r < 2,$$

*and*

$$\|u(t) - v(t)\|_2 \le C t^{-\frac{n}{4}-\frac{1}{2}} \ln(t+1), \qquad \text{for } r = 2,$$

*where $u$ and $v$ are solutions for (NSE) and for (SE), respectively.*

Ukai [19] and Borchers and Miyakawa [6] studied the decay problem for the Stokes equations (1.3)-(1.2). We review the Ukai formula for (SE).

Denote by the Riesz' operators, $R_j$, $j = 1, \dots, n$, and $S_j$, $j = 1, \dots, n-1$, which are the singular integral operators with the symbols

$$\sigma(R_j) = i\xi_j/|\xi|, \quad j = 1, \dots, n, \qquad \text{and} \qquad \sigma(S_j) = i\xi_j/|\bar{\xi}|, \quad j = 1, \dots, n-1,$$

where $\xi = (\xi_1, \dots, \xi_n) = (\bar{\xi}, \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ is the dual variable to $x \in \mathbb{R}^n$. Set

$$\bar{R} = (R_1, R_2, \cdots, R_{n-1}), \qquad S = (S_1, S_2, \cdots, S_{n-1}),$$

and define the operators $V_1$ and $V_2$ by

$$V_1 u_0 = -S \cdot \bar{u}_0 + u_{0,n}, \qquad V_2 u_0 = \bar{u}_0 + S u_{0,n},$$

where $u_0 = (u_{0,1}, u_{0,2}, \dots, u_{0,n}) = (\bar{u}_0, u_{0,n})$. Furthermore, let $h$ be the restriction operator from $\mathbb{R}^n$ to $\mathbb{R}^n_+$, that is, $hf = f|_{\mathbb{R}^n_+}$, and $e$ the extension operator from $\mathbb{R}^n_+$

over $\mathbb{R}^n$ with value 0: $ef = f$ for $x_n > 0$ and $ef = 0$ for $x_n < 0$. We also define the operator $U$ by

$$Uf = h\bar{R} \cdot S(\bar{R} \cdot S + R_n)ef.$$

By Ukai [19], solution $u$ of (1.3) is represented as

$$\begin{aligned} u_n &= UE(t)V_1u_0, \\ \bar{u} &= E(t)V_2u_0 - SUE(t)V_1u_0, \end{aligned}$$

where $\gamma$ is the trace operator to the boundary $\partial\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \{0\}$, $\gamma u = u(x', 0, t)$ if $u \in C(\overline{\mathbb{R}^n_+})$, and

$$E(t)f = \int_{\mathbb{R}^n_+} \big(K(\bar{x} - \bar{y}, x_n - y_n, t) - K(\bar{x} - \bar{y}, x_n + y_n, t)\big)\, f(y)\, dy.$$

The decay rates are given in Bae and Choe [2].

**Theorem 4.3.** *Let $u$ be the solution of the Stokes equation* (1.3) *with initial data $u_0 \in L^2_\sigma$. Then, we have that for $1 < r \le q < \infty$*

$$\left(\int_{\mathbb{R}^n_+} |u(x,t)|^q\, dx\right)^{1/q} \le \frac{C}{t^{\frac{n}{2}(\frac{1}{r} - \frac{1}{q}) + \frac{1}{2}}} \left(\int_0^\infty \int_{\mathbb{R}^{n-1}} |y_n|^r |u_0(y)|^r\, d\bar{y}\, dy_n\right)^{1/r}.$$

*For $q = \infty$ we have that*

$$\|u\|_\infty \le Ct^{-\frac{n}{2r} - \frac{1}{2}} \|y_n u_0(y)\|_r$$

*for $1 < r$.*

In Giga, Matsui and Shimizu [8], they rewrote Ukai's formula. Define the operator $\Lambda$ with symbol $\sigma(\Lambda) = |\bar{\xi}|$. Then, the solution $u$ of the Stokes equation (1.1) is given in [8] as the restriction $hv$ of a vector field $v \overset{def}{=} (\bar{v}, v_n)$ of the form

$$\begin{aligned} v_n &= \bar{R} \cdot S(\bar{R} \cdot S + R_n)eE(t)V_1u_0, \\ \bar{v} &= E(t)V_2u_0 - S\bar{R} \cdot S(\bar{R} \cdot S + R_n)eE(t)V_1u_0. \end{aligned} \tag{4.1}$$

The following lemma is given in [8].

**Lemma 4.4.** *Let $j$ be an integer with $1 \le j \le n$. Assume that $\nabla \cdot u_0 = 0$ in $\mathbb{R}^n_+$. Then, the first spatial derivative of $v$ are expressed as*

$$\begin{aligned} \partial_j v_n &= -R_j\big[\bar{R} \cdot \Lambda eE(t)\overline{u_0} - R_n\bar{\nabla} \cdot eE(t)\overline{u_0} + \bar{R} \cdot \bar{\nabla} eE(t)u_{0,n} + R_n\Lambda eE(t)u_{0,n}\big], \\ \partial_j \bar{v} &= \partial_j E(t)\overline{u_0} + w_j + R_j\big[\bar{R}(\bar{\nabla} \cdot eE(t)\overline{u_0}) - R_n\bar{\nabla}(\bar{\nabla}\Lambda^{-1} \cdot eE(t)\overline{u_0}) \\ &\qquad - \bar{R}\Lambda eE(t)u_{0,n} + R_n\bar{\nabla} eE(t)u_{0,n}\big], \end{aligned}$$

*where*

$$w_j = \begin{cases} \partial_j \bar{\nabla}\Lambda^{-1}E(t)u_{0,n} & \text{for } 1 \le j \le n-1, \\ -\bar{\nabla}\big(\bar{\nabla} \cdot \Lambda^{-1}F(t)\overline{u_0}\big) & \text{for } j = n, \end{cases}$$

*and $u_0 = (\overline{u_0}, u_{0,n}) = (u_{0,1}, \cdots, u_{0,n})$. Here, $F(t)f$ is the solution of the heat equation in $\mathbb{R}^n_+$ with Neumann data $\partial_n F(t)f|_{x_n=0} = 0$ and the initial data $f$:*

$$[F(t)f](x) \overset{def}{=} \int_{\mathbb{R}^n_+} \big(K(\bar{x} - \bar{y}, x_n - y_n, t) + K(\bar{x} - \bar{y}, x_n + y_n, t)\big)f(y)\, dy.$$

The decay rates in the Hardy space $\mathcal{H}^1$ are given in Bae [1].

**Theorem 4.5.** *Assume that $\nabla \cdot u_0 = 0$ in $\mathbb{R}^n_+$. Let $u$ be the solution of the Stokes equation* (1.1) *with initial data $u_0$. Then, there is a constant $C$ independent of $u_0$ such that*

$$\|\nabla u(t)\|_{\mathcal{H}^1(\mathbb{R}^n_+)} \le Ct^{-1} \int_{\mathbb{R}^n_+} y_n |u_0(y)|\, dy.$$

Since

$$\|\nabla u\|_{L^1(\mathbb{R}^n_+)} \le \|\nabla u\|_{\mathcal{H}^1(\mathbb{R}^n_+)} \le \|\nabla v\|_{\mathcal{H}^1} \le Ct^{-1}\|y_n u_0\|_{L^1(\mathbb{R}^n_+)},$$

we have the following.

**Corollary 4.6.** *Assume that $\nabla \cdot u_0 = 0$ in $\mathbb{R}^n_+$. Let $u$ be the solution of the Stokes equation* (1.1) *with initial data $u_0$. Then, there is a constant $C$ independent of $u_0$ such that*

$$\|\nabla u(t)\|_{L^1(\mathbb{R}^n_+)} \le Ct^{-1} \int_{\mathbb{R}^n_+} y_n |u_0(y)|\, dy.$$

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