Trends in Mathematics
Information Center for Mathematical Sciences
Volume 4, Number 1, June 2001, Pages 21–25

DOMAINS OF HOLOMORPHY AND AUTOMORPHISMS

KANG-TAE KIM

Dedicated to Professor Kyong-Taik Hahn on his retirement

ABSTRACT. I try to introduce one of the important themes and trends of researches concerning the geometric study of bounded pseudoconvex domains in $\mathbb{C}^n$ and their automorphism groups. As this article is prepared for the readers who are not experts in the field, I have taken the liberty of casually rephrasing several important theorems in order to avoid all the technicalities, thinking that they may be shunning the readers off, at the risk of possible misrepresentation — if it did happen, I ask for pardons from the experts, especially from the authors of theorems which I cite in this article.

One of the milestones in the study of complex analytic functions and mappings in several complex variables is the seminal theorem of H. Poincarè [1] which demonstrates that

> *Riemann's mapping theorem cannot be directly generalized to any complex dimension higher than one.*

Let us take a moment to appreciate how Poincarè justified this statement. Take two domains in the complex two dimensional Euclidean space $\mathbb{C}^2$: the ball

$$B^2 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 < 1\}$$

and the bi-disc (meaning, the product of two discs)

$$D^2 = \{(z,w) \in \mathbb{C}^2 \mid |z| < 1, |w| < 1\}.$$

Expecting a contradiction, suppose that there exists a biholomorphism (= a bijective holomorphic mapping) $f : B^2 \to D^2$. Composing it with a Möbius transform of D^2 we may assume without loss of generality that $f(0,0) = (0,0)$. Next, consider the sets of holomorphic automorphisms of these domains, denoted by $\operatorname{Aut} B^2$ and $\operatorname{Aut} D^2$ respectively. Each of them turns out to be a topological group under the law of composition and with respect to the topology of uniform convergence on compact subsets. Thus we call them *automorphism groups.* Now, the map

2000 *Mathematics Subject Classification.* 32Q45.
Key words and phrases. Domains of holomorphy, Automorphisms Groups, Invariant metrics.
Research of the author is supported in part by KOSEF Interdisciplinary Research Program Grant 1999-2-102-003-5 and The BK-21 Program. The manuscript of this article was prepared during the author's visiting research period at The American Institute of Mathematics in August 2000.
Received August 30, 2000.

[1]See for instance "S.G. Krantz, Function theory of several complex variables, Wadsworth & Brooks/Cole, 1992.

$f^* : \operatorname{Aut} B^2 \to \operatorname{Aut} D^2$ defined by $f^*(\sigma) = f \circ \sigma \circ f^{-1}$ turns out to be a topological group isomorphism. In particular, this map preserves the identity components of the isotropy subgroups at the origin. Now, a straightforward argument verifies that the identity component of isotropy subgroup for the bi-disc at the origin consists only of the rotations of the form $(z, w) \mapsto (e^{i\theta}z, e^{i\psi}w)$ where $\theta, \psi \in \mathbb{R}$, which obviously constitute an abelian group. On the other hand, the connected component of isotropy subgroup for the ball contains the complex two dimensional unitary transformations, resulting in that it is non-abelian. Since abelian groups cannot be isomorphic to a non-abelian group, we now have reached at the desired contradiction.

Although several other proofs (some are simpler than the preceding argument) to the above mentioned theorem of Poincarè are known by now, this original proof still maintains its own merits: for example, it shows in particular that the automorphism group of a given domain in $\mathbb{C}^n$ is a biholomorphic invariant. This, however, is not the end but only a beginning; other natural questions arise immediately. Here are a few.

> *How many different groups may arise as automorphism groups of a given domain? Among possible automorphism groups, which are more special than the others? For instance, can the automorphism group of a domain actually determine the domain up to biholomorphic equivalence?*

There are many more questions one may ask, as this subject is of an active and current research interest. But let us limit ourselves in this article to the preceding questions for the time being. For a start, I would like to cite the theorem of H. Cartan [2] which states that

> *the automorphism group of every bounded domain in $\mathbb{C}^n$ is in fact a Lie group.*

So, we may now ask which Lie groups may actually arise as the automorphism groups of bounded domains. A result of E. Bedford and J. Dadok[3] (also by R. Saerens and W. Zame)[4] says that *every compact Lie group can be realized as the automorphism group of a bounded domain of some $\mathbb{C}^n$.*

At this point, one might like to think that there may be some good correspondences (such as a finite-to-one map etc.) between the bounded domains and its automorphisms. But a remarkable theorem by D. Burns, S. Shnider and R.O. Wells [5] says otherwise:

> *In an arbitrarily small smooth perturbation of the unit ball in $\mathbb{C}^n$ for every $n > 1$, there is an infinite dimensional family of biholomorphically distinct domains. Furthermore, all these domains can be selected so that their automorphism groups consist of the identity map only!*

It turned out that one can do more in this direction, which is a theorem by R.E. Greene and S.G. Krantz [6]. In order to state their theorem, as well as to clarify

[2] *Ibid.*
[3] E. Bedford and J. Dadok, Comm. Math. Helv., 1987.
[4] R. Saerens and W. Zame, Trans. Amer. Math. Soc.
[5] D. Burns, S. Shnider and R.O. Wells, Jr., Invent. Math., 1978.
[6] R.E. Greene and S.G. Krantz, Adv. in Math., 1982.

the rough statement of preceding theorem, we define the concept of C^∞*-closeness* between bounded domains.

Consider two bounded domains G_1, G_2 in $\mathbb{C}^n$. We will say that two such domains are *comparable*, if there exists a C^∞ diffeomorphism, say F of a neighborhood of the closure of G_1 onto a neighborhood of the closure of G_2 such that $F(G_1) = G_2$. Denote by $\mathsf{Diff}(G_1, G_2)$ Then, we will consider

$$d_k(G_1, G_2) = \inf_{F \in \mathsf{Diff}(G_1, G_2)} \sum_{j=0}^{k} \sup_{x \in G_1} \left\| D^k (F - I)(x) \right\|$$

where: D^k denotes all possible derivatives of all components of the mapping in consideration, the norm is the sum of absolute values of all possible such terms, and finally I stands for the identity mapping. In case we restrict ourselves only to contractible bounded domains in $\mathbb{C}^n$ with a C^∞ smooth boundary, and call this collection $\mathcal{S}_n$, then d_k $(k = 0, 1, 2, \dots)$ defines a sensible topology on $\mathcal{S}_n$. In this way, we may even metrize the space $\mathcal{S}_n$. Thus, we have introduced a concept of closeness of two domains, at least for the contractible bounded domains with smooth boundaries in $\mathbb{C}^n$.

It should be obvious to a reader that this concept of distance on the space $\mathcal{S}_n$ gives a better formulation of the aforementioned theorem of Burns-Shnider-Wells. Now, we state part of theorem on automorphisms by R.E. Greene and S.G. Krantz.

> *In the metric space $\mathcal{S}_n$ for each $n > 1$, the subset of elements with their automorphism group compact is open dense (i.e. generic).*

Greene and Krantz has also demonstrated that, for every $\epsilon > 0$, the ϵ-neighborhood of any element of $\mathcal{S}_n$ $(n > 1)$ contains an infinite dimensional family of holomorphically inequivalent elements whose automorphism groups are compact and isomorphic to each other. In other words, a bounded domains with a smooth boundary is "flexible" in the sense that it can be deformed, while keeping their automorphism groups unchanged. (This turns out to be quite important and useful for various purposes. There is also a comparable results in Differential Geometry.)

Perhaps, we are sufficiently satisfied with the above results concerning the domains with noncompact automorphism groups. Then, what happens to *bounded domains with their automorphism groups non-compact?*

It turns out that the bounded domains with a noncompact automorphism group are very rigid. As the first celebrated result which verifies such a claim, I would like to recite is the following theorem of B. Wong [7]:

> *Every bounded strongly pseudoconvex domain $\mathbb{C}^n$ with a noncompact automorphism group is biholomorphic to the unit ball in $\mathbb{C}^n$.*

The terminology "strong pseudoconvexity" is the concept that the boundary can be transformed to a strongly convex set (i.e. the normal curvature of the boundary surface is positive in all tangential directions everywhere) by a suitable holomorphic change of local coordinate system centered at each boundary point. Such domains

[7] B. Wong, Invent. Math. 1977.

turns out to be one of the most standard domains [8] in the study of complex analysis in higher dimensions. [9]

Then, the clever modification and localization by J.P. Rosay [10] of Wong's theorem should be mentioned: in case the automorphism group of a bounded domain is non-compact, this localization implies that a certain sequence of automorphisms must carry an interior point successively to a boundary point. (This is a consequence of Montel's theorem and a standard argument on topological group actions.) Rosay's contribution, among other things, is that Wong's theorem is valid if a bounded domain admits a strongly pseudoconvex boundary point (notice that this concept is purely local!) at which an automorphism orbit of an interior point accumulates. Notice that Rosay's version now implies: *if a bounded domain with a globally C^2 smooth boundary is a holomorphic covering space of a compact complex variety, the domain is biholomorphic to the unit ball.* (This answers one of the old questions posed around 1940.)

It became clear in Wong's work (as well as in the localized version of Rosay) that the boundary behavior of biholomorphic invariants (in Wong's case, the ratio of Kobayashi-Eisenmann and Carathéodory volume elements) plays a key role. Indeed, other approaches (including more differential geometric ones) in this direction are resonant: the study of boundary behavior of the Bergman curvature near the strongly pseudoconvex boundary by P. Klembeck, K.T. Kim and J. Yu [11]) shows that near the strongly pseudoconvex boundary point the Bergman curvature converges asymptotically to the curvature tensor of the unit ball, for which all the holomorphic sectional curvature is negative constant. Since biholomorphic mappings are Bergman isometries, this shows through a suitable normal family trick that the Bergman metric has negative constant holomorphic curvature everywhere. Then, Q.K. Lu's work [12] concludes that the domain is biholomorphic to the unit ball equipped with the Poincarè-Bergman metric!

Boundary behavior of holomorphic invariants are still of a current interest. The widely known work by I. Graham [13] which investigated the boundary behavior of the Kobayashi metric has been generalized in various different contexts. The latest work (hence the best treatment so far) in this line seems the recent work by Sunhong Lee, whose survey article in this direction appears in this same volume. There are many more questions concerning the other holomorphic invariants which one can explore in this direction.

Returning to the study of domains with noncompact automorphism groups, there has appeared a more "modern" method — nowadays called *the scaling method.* This may have a deeper root, as most mathematical ideas do. But I feel that it is fair to say that S. Pinchuk [14] is the modern founder. The main idea is as follows:

---

[8]See S.G. Krantz, *Op Cit.*

[9]The most natural class is the class of domains of existence for holomorphic functions, i.e. domains possess holomorphic functions which cannot be extended to a larger domain. Such domains are called *domains of holomorphy. It is known that every domain of holomorphy is can be exhausted by bounded strongly pseudoconvex domains. (E.E. Levi)*

[10]J.P. Rosay, Ann. Inst. Fourier (Grenoble), 1979

[11]P. Klembeck, Indiana U. Math. J. 1978.; K.T. Kim and J. Yu, Pacific J. Math., 1996.

[12]Qi-Keng Lu (=K.H. Look), Chinese Math., 1966

[13]I. Graham, Trans. Amer. Math. Soc. 1975.

[14]See the following founding articles of the scaling technique: S. Pinchuk, Math. USSR Sbornik, 1981. See also, S. Frankel, Acta Math. 1989, and K.T. Kim, Trans. Amer. Math. Soc. 1990 and Lecture Notes Series 13, GARC of Seoul National University

Let $f_j \in \operatorname{Aut} G$ and $q \in G$ be such that $f_j(q)$ accumulates at a boundary point. Then, f_j tends to shrink any arbitrary big compact subset of G to a very small set near the accumulation boundary point of the orbit $\{f_j(q)\}_{j=1}^{\infty}$. Thus, find a divergent sequence of complex affine maps L_j such that $L_j \circ f_j$ becomes a normal family whose subsequential limits are biholomorphisms.

This simple idea has added so much research prowess to the field, because the local Hausdorff set limit of a suitable subsequence of sets $L_j(G)$ (Note that $f_j(G) = G$!) has often yielded relatively simple domains which one may easily recognize. This technique in particular gives another (more intuitive) proof of Wong-Rosay Theorem. Then, with techniques using a version of dynamics involving holomorphic vector fields, E. Bedford and S. Pinchuk obtained recently [15] the following remarkable theorem:

Every bounded domain in $\mathbb{C}^2$ with a real analytic boundary possesses a noncompact automorphism group if and only if it is biholomorphic to the domain $\{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^{2m} < 1\}$, for some positive integer m.

This theorem covers the category of domains with non-flat boundaries in the sense that no local holomorphic coordinate change can transform the boundary surface to a Levi-flat surface, and the obstruction resides in a finite order complex jet. For the Levi flat case, there is this theorem by K.T. Kim and A. Pagano[16] :

The holomorphic universal covering space of every bounded analytic polyhedron in $\mathbb{C}^2$ with no singularities other than normal crossings in its boundary which possesses a noncompact automorphism group is biholomorphic to the bi-disc.

The typical techniques in the proof of this theorem consists of the scaling method, the Hermitian symmetry, the Wu invariant metric [17]. I hope all these show that, although the scaling method appears in some sense primitive, it can be quite effective when combined with some suitable methods and ideas.

There are several other important directions and results this essay has failed to cover: the seminal work of E. Cartan on bounded symmetric domains, the Greene-Krantz conjecture which is of a current interest, the characterization of the Hilbert ball (in the infinite dimensions) by its automorphism group obtained recently by Kim and Krantz, and so forth. However, despite such shortcomings, I wish that this article somehow helped the readers to be more acquainted with this stream of research.

Pohang University of Science and Technology, Pohang 790-784 Korea
E-mail address: kimkt@postech.ac.kr

[15] E. Bedford and S. Pinchuk, Preprint.

[16] K.T. Kim and A. Pagano, To appear in J. Geom. Analysis; See also K.T. Kim, Math. Ann., 1992.

[17] H. Wu, Math. Notes, 1993.