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A SURVEY ON THE REGULARITY OF THE INTERFACES FOR SOME NONLINEAR PARABOLIC EQUATIONS

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Abstract. In this article, we survey the regularity of the interfaces for the porous medium equation, parabolic p-Laplacian equation and some doubly degenerate parabolic nonlinear equation.

1. Introduction

We consider the Cauchy problem of the form

$$u_t = \operatorname{div}\left(|\nabla u^m|^{p-2}\nabla u^m\right) \quad \text{in} \quad \mathcal{R}^n \times (0,\infty), \tag{1.1}$$

$$u(x,0) = u_0 \quad \text{on} \quad \mathcal{R}^n. \tag{1.2}$$

Here we suppose that $m > 0$, $p > 1 + 1/m$ and that u_0 is a nonnegative continuous function with compact support.

Equation (1.1) appears in a number of applications to describe the evolution of diffusion processes, in particular the flow of non-Newtonian fluids in a porous medium in which u stands for the density [14].

There are, in general, no classical solutions of (1.1) and (1.2) because of the degeneracy of the equation at points where $\nabla u = 0$ or $u = 0$. Nevertheless, it is well-known [14] that the Cauchy problem has a weak solution $u \in C^\alpha$.

Solutions of the Cauchy problem (1.1) and (1.2) have an important special property: the finite propagation speed. If u_0 has compact support, so does $u(.,t)$ for each positive time t. Therefore there is an interface Γ that seperates the region where $u > 0$ from the region where $u = 0$. If $p = 2$ in equation (1.1), then it becomes the porous medium equation

$$u_t = \Delta u^m \quad \text{in} \quad \mathcal{R}^n \times (0,\infty), \tag{1.3}$$

and if $m = 1$ then it reduces to the parabolic p-Laplacian equation

$$u_t = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \quad \text{in} \quad \mathcal{R}^n \times (0,\infty). \tag{1.4}$$

The regularity of the interfaces for the above three equations were studied by many authors. In this article we introduce some recent results, especially on the work done by Daskalopoulos and Hamilton [9].

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2. POROUS MEDIUM EQUATION

About the regularity of the interfaces for the above three equations, the porous medium equation has the longest history and the most fruitful results.

Caffarelli and Friedman ([4], [5]) showed that the interface can always be described by a Hölder continuous function $t = S(x)$, $x \in \text{supp} u_0$, for any initial data. In one-dimensional case much more is known : It has been shown in [15] and [4] that if the support of the initial data is an interval, the interface consists of two Lipschitz continuous curves. Moreover there exists waiting times so that each interface does not move until the waiting time and after that they are C^1. However at the waiting time the interface may have a corner. Therefore Lipschitz is the optimal regularity [2]. Aronson and Vázquez([3]) and independently Höllig and Kreiss([10]) showed that after the waiting time the interfaces are smooth. Moreover Angenent([1]) showed the interfaces are real analytic after the wating times.

In dimensions $n \geq 2$, Caffarelli, Vázquez and Wolanski([7]) showed that, under some nondegeneracy conditions on the initial data, the interface can be described by a Lipschitz continuous function after a large time. Caffrelli and Wolanski([8]) improved this result, showing that, under the same hypotheses, the interface is a $C^{1,\alpha}$ surface after a large time. Recently [19], we showed that the interface can be represented as a C^∞ surface after a large time if the solution is radial symmetry. The main purpose of this survey is to introduce the work of Daskalopoulos and Hamilton([9]). By assuming that the initial data u_0 is strictly positive in the interior of a compact domain Ω in $\mathcal{R}^2$, with $u_0 = 0$ on $\partial\Omega \cup (\mathcal{R}^2 \setminus \Omega)$, and denoting by d the distance to the boundary of Ω, they showed

Theorem 2.1. *If u_0, Du_0 and dD^2u_0, restricted to the compact domain Ω, extend continuously up to $\partial\Omega$, with expressions which are Hölder continuous on Ω of class C^α, for some $\alpha > 0$ and $Du_0 \neq 0$ along $\partial\Omega$, then there exists a number $T > 0$ for which the initial value problem*

$$\begin{cases} u_t = u\Delta u + \frac{1}{m-1}|Du|^2, & (x,t) \in \mathcal{R}^n \times [0,T], \\ u(x,0) = u_0, & x \in \mathcal{R}^n, \end{cases}$$

admits a solution u which is smooth up to the interface Γ, when $0 < t < T$. In particular the free boundary Γ is a smooth surface when $0 < t < T$.

In showing the Theorem 2.1, they first study the following linear degenerate equation

$$u_t = x(u_{xx} + u_{yy}) + \nu u_x + g \tag{2.1}$$

with $\nu > 0$ on $\mathcal{S}_0$= half-space : $x \geq 0$. The diffusion in this equation is governed by a Riemannian metric

$$ds^2 = \frac{dx^2 + dy^2}{2x}.$$

By using this new metric, we define the Hölder semi-norm

$$||g||_{H_s^\alpha}(\mathcal{A}) = \sup_{P_1 \neq P_2} |g(P_1) - g(P_2)|/s[P_1, P_2]^\alpha$$

and the norm

$$||g||_{C_s^\alpha}(\mathcal{A}) = ||g||_{C^0}(\mathcal{A}) + ||g||_{H_s^\alpha}(\mathcal{A}).$$

Here $C_s^\alpha(\mathcal{A})$ a Banach space of Hölder continuous functions on $\mathcal{A}$ with respect to the metric s and $s[P_1, P_2]$ is the parabolic distance between P_1 and P_2 (for the detail see[9]). If the operator L_0 is defined by

$$L_0 = u_t - x(u_{xx} + u_{yy}) - \nu u_x$$

then it is a continuous linear map from $C_s^{2+\alpha}$ into C_s^α. As usual for each nonnegative integer k, we can extend these concepts to spaces of higher order derivatives and hence the operator

$$L_0 : C_s^{k,2+\alpha}(\mathcal{A}) \to C_s^{k,\alpha}(\mathcal{A})$$

defines a continuous linear map. Let $\mathcal{S} = \mathcal{S}_0 \times [0, \infty)$ and $\mathcal{S}_T = \mathcal{S}_0 \times [0, T]$. Then we have

Theorem 2.2. *Assume that $g \in C_s^{k,\alpha}(\mathcal{S})$ and $u_0 \in C_s^{k,2+\alpha}(\mathcal{S})$, with both g and u_0 compactly supported in $\mathcal{S}$ and $\mathcal{S}_0$ respectively. Then for any $\nu > 0$ and $T > 0$, the initial value problem*

$$\begin{cases} L_0 u = g & in \quad \mathcal{S}, \\ u(x, 0) = u_0, & on \quad \mathcal{S}_0 \end{cases}$$

admits a unique solution $u \in C_s^{k,2+\alpha}(\mathcal{S}_T)$. Moreover

$$||u||_{C_s^{k,2+\alpha}(\mathcal{S}_T)} \leq C(T) \left(||u_0||_{C_s^{k,2+\alpha}(\mathcal{S}_0)} + ||g||_{C_s^{k,\alpha}(\mathcal{S})} \right)$$

for some constant $C(T)$ depending only on α, k, ν and T.

Theorem 2.2 is proved by establishing several Scauder type estimates, barrier functions and parabolic theories.

Now consider the following linear degenerate equations of the form

$$\omega_t = (\theta a^{ij} \omega_{ij} + b^i \omega_i + c\omega) + g \tag{2.2}$$

on the cylinder $\Omega \times [0, \infty)$, where Ω is a compact domain in $\mathcal{R}^2$ with smooth boundary and $i, j \in \{x, y\}$. We assume the coefficient matrix (a^{ij}) is strictly positive and all coefficients belong to some Hölder spaces. θ which is a function carrying the degeneracy is assumed to be smooth on Ω and strictly positive in its interior, with

$$\theta(P) = \text{dist}(P, \partial\Omega)$$

for all $P \in \Omega$ sufficiently close to $\partial\Omega$. Now denoting by L the oprator

$$L\omega = \omega_t - (\theta a^{ij} \omega_{ij} + b^i \omega_i + c\omega),$$

by Ω_σ, for $\sigma > 0$, the set

$$\Omega_\sigma = \{x \in \Omega \ : \ \text{dist}(x, \partial\Omega) \geq \sigma\},$$

by Q_T, for $T > 0$, the cylinder $\Omega \times [0, T]$ and as usual, by $n = (n_1, n_2)$ the interior unit normal to $\partial\Omega$, Daskalopoulos and Hamilton extended their results and obtained

Theorem 2.3. *Let Ω be a compact domain in $\mathcal{R}^2$ with smooth boundary and let k be a nonnegative integer, $a \in (0, 1)$ and $T > 0$. Assume that the coefficients a^{ij}, b^i and c belongs to $C_s^{k,\alpha}(Q_T)$ and satisfies*

$$a^{ij} \xi_I \xi_j \geq \lambda |\xi|^2 > 0 \quad \xi \in \mathcal{R}^2 \setminus \{0\}$$

and

$$||a^{ij}||_{C_s^{k,\alpha}(Q_T)} \quad ||b^i||_{C_s^{k,\alpha}(Q_T)} \quad ||c||_{C_s^{k,\alpha}(Q_T)} \leq 1/\lambda$$

and

$$b^i n_i \geq \nu > 0 \quad on \quad \partial\Omega \times [0, T]$$

for some positive constants λ *and* ν. *In addition, assume that* θ *is a smooth function* Ω, *strictly positive in its interior, with* $||\theta||_{C^\infty(\Omega)} \leq 1$ *and such that*

$$\theta(P) = dist(P, \partial\Omega), \qquad P \in \Omega \setminus \Omega_\sigma$$

for some $\sigma > 0$. *Then for any* $\omega_0 \in C_s^{k,2+\alpha}(\Omega)$ *and* $g \in C_s^{k,\alpha}(Q_T)$ *there exists a unique solution* $\omega \in C_s^{k,2+\alpha}(Q_T)$ *of the initial value problem*

$$\begin{cases} L\omega = g & in \quad Q_T, \\ \omega(x,0) = \omega_0(x), & on \quad \Omega \end{cases}$$

satisfying

$$||\omega||_{C_s^{k,2+\alpha}(Q_T)} \leq C(T) \left(||\omega_0||_{C_s^{k,2+\alpha}(\Omega)} + ||g||_{C_s^{k,\alpha}(Q_T)} \right).$$

The constant $C(T)$ *depends only on the domain* Ω *and the numbers* $\alpha, k, \lambda, \nu, \sigma$ *and* T.

Next consider the following quasilinear degenerate equations of the form

$$\omega_t = \theta F^{ij}(t, x, y, \omega, D\omega)\omega_{ij} + G(t, x, y, \omega, D\omega)$$

on $Q_T = \Omega \times [0, T]$. Define operators P and M by

$$P\omega = \theta F^{ij}(t, x, y, \omega, D\omega)\omega_{ij} + G(t, x, y, \omega, D\omega)$$

and

$$M\omega = \omega_t - P\omega.$$

Then we have the following theorem :

Theorem 2.4. *Let* $\omega_0 \in C_s^{k,2+\alpha}(\Omega)$ *and assume the linearization* $DM(\bar{\omega})$ *of* $M\omega$ *defined on* Q_T *satisfies the hypotheses of Theorem 2.3 at all points* $\bar{\omega} \in C_s^{k,2+\alpha}(Q_T)$, *such that* $||\bar{\omega} - \omega_0||_{C_s^{k,2+\alpha}(Q_T)} \leq \mu$, $\mu > 0$. *Then, there exists a number* $\tau_0 \in (0, T]$ *depending only on* α, k, λ, ν *and* μ, *for which the initial value problem*

$$\begin{cases} \omega_t = \theta F^{ij}(t, x, y, \omega, D\omega)\omega_{ij} + G(t, x, y, \omega, D\omega) & in \quad \Omega \times [0, \tau_0], \\ \omega(x,0) = \omega_0(x), & on \quad \Omega \end{cases}$$

admits a solution $\omega \in C_s^{k,2+\alpha}(\Omega \times [0, \tau_0])$. *Moreover*

$$||\omega||_{C_s^{k,2+\alpha}(\Omega \times [0,\tau_0])} \leq C||\omega_0||_{C_s^{k,2+\alpha}(\Omega)}$$

for some positive constant C *which depends only on* α, k, λ, ν *and* σ.

Now applying all the results in this section, we have

Theorem 2.5. *Let* Ω *be a compact domain in* $\mathcal{R}^2$ *and* $u_0 \in C_s^{2+\alpha}(\Omega)$, *for some* $\alpha \in (0, 1)$, *with* $u_0 = 0$ *at* $\partial\Omega$ *and* $u_0 > 0$ *in the interior of* Ω. *Assume that*

$$Du_0(x, y) \neq 0, \quad (x, y) \in \Omega.$$

Then there exists a number $T > 0$, *for which the free boundary problem*

$$\begin{cases} u_t = u\Delta u + \frac{1}{m-1}|Du|^2, & (x, y, t) \in \Omega_t \times [0, T], \\ u(x, y, 0) = u_0, & (x, y) \in \Omega, \end{cases}$$

admits a solution u *which is smooth up to the free boundary. In particular the interface* $\partial\Omega_t \times (0, T]$ *is smooth.*

Here Ω_t is the closure of the set

$$\{(x, y) \in \mathcal{R}^2 \ : \ u(x, y, t) > 0\}.$$

The since $C^\alpha(\Omega) \subset C_s^\alpha(\Omega)$, for all $\alpha > 0$, we have Theorem 2.1.

3. parabolic p-Laplacian equation (1.4) and equation (1.1)

For the equation (1.4), the first remarkable result for the regularity of the interface, for the general space dimensions, can be found in [6]. They showed the interface can be, under some nondegeneracy conditions on the initial data, represented as a Lipschitz surface after a large time and we [16] extended this reult by showing the interface can be expressed as a $C^{1,\alpha}$ surface after a large time. By applying the methods in [9], we [12] showed the interface is smooth for $t \in (0, T]$ for some constant $T > 0$. For the one dimensional case, we [11] showed that the interfaces are smooth after the waiting time.

For the equation (1.1), for the dimensions ≥ 2, after Zhao [21] showed that the interface is Lipschitz, we [17] extended the result by showing the interface can be expressed as a $C^{1,\alpha}$ surface after a large time. For $0 < t \leq T$ for some constant $T > 0$, as in the equation (1.4), we [18] showed that the interfaces is smooth. For the one dimensional case, we [13] showed that the interfaces are smooth after the waiting times.

Now we suggest the following problem :

Problem *Can the interface of the porous medium equation be a smooth surface after a large time ?*

***Remark* 3.1.** *Since the equations (1.1) and (1.4) are, at least formally, quite similar to the porous medium equation, once we answer the above problem, we can expect the interfaces for the equations (1.1) and (1.4) can be represented as smooth surfaces after a large time.*

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