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CARTAN-FUBINI TYPE EXTENSION OF HOLOMORPHIC MAPS

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ABSTRACT. In this article, we will discuss a recent joint work with N. Mok on the extension of holomorphic maps which preserves the set of tangent directions to minimal rational curves on Fano manifolds with second Betti number 1. Some historical background and motivation for such study is explained. The result has an application to the rigidity of holomorphic maps whose targets are Fano manifolds where such extension of holomorphic maps holds.

One of the basic questions in differential geometry is the problem of congruence. The oldest result of this type is about curves in the Euclidean plane. Let $C, C' \subset \mathbf{E}^2$ be two pieces of curves in the plane. When does there exist a Euclidean motion F of the plane such that $F(C) = C'$? The problem can be stated in a more refined manner as: given a diffeomorphism $f : C \to C'$, when can f be extended to a Euclidean motion? There exists an easy answer to this question. Choosing unit speed parametrization, we can compute the curvature κ, κ' of the curves. Then f can be extended to a Euclidean motion if and only if it preserves the curvature, i.e. $f^*\kappa' = \kappa$. This result is very classical and I don't know whom to attribute it to (Huygens, Leibniz, or Newton ?).

The congruence problem for surfaces in $\mathbf{E}^3$ was solved by Bonnet (1867, [Sp] III p.79. or [ON] VI.8.3). Given a piece of surface $S \subset \mathbf{E}^3$, we introduce the first fundamental form, which is the restriction of the Euclidean metric to S, and the second fundamental form, which measures how the unit normal vector is moving along the surface (in [ON], the second fundamental form is called 'the shape operator'). Bonnet's result says that if a diffeomorphism $f : S \to S'$ between two pieces of surfaces in $\mathbf{E}^3$ preserves the first and the second fundamental forms, then it can be extended to a Euclidean motion of $\mathbf{E}^3$. The result can be generalized to hypersurfaces in higher dimensional Euclidean spaces: a diffeomorphism preserving the first and the second fundamental forms of pieces of hypersurfaces can be extended to a Euclidean motion ([Sp] IV p.64). Undoubtedly, these results are true gems of nineteenth century differential geometry.

Through the nineteenth century, big developments were made and new objects were introduced which changed the perspective of geometry great deal. Let me mention three of them which are closely related to our discussion. First of all, through the works of Gauss for surfaces and Riemann for higher dimensions, the importance of intrinsic geometry of the manifolds was realized. From this 'modern'

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point view, the properties of the manifold which are independent of its embedding in the Euclidean space are more important than the extrinsic properties. The first fundamental form is an intrinsic geometric object, called 'Riemannian metric' in today's terms, but the second fundamental form is purely extrinsic object depending on how the manifold is embedded. Secondly, through the explosive development of complex analysis in the early nineteenth century, the study of complex manifolds and holomorphic maps became an important area of mathematics, especially because of its connection with algebraic geometry as pioneered by Riemann. Thirdly, starting from the work of Poncelet, the projective space was studied extensively by many geometers and projective geometry became one of the major branch of geometry. As a consequence, the projective space attains a status as important as the Euclidean space. Keeping in mind the last two points, one may ask the following generalization of Bonnet's result.

Problem *Let S, S' be two pieces of complex hypersurfaces in the complex projective space $\mathbf{P}^{n+1}$. Let $f : S \to S'$ be a biholomorphic map. When can we extend f to a projective transformation of $\mathbf{P}^{n+1}$?*

This question was studied by G. Fubini around 1920. In projective geometry, the first fundamental form does not make sense because the notion of metric does not exist. The second fundamental form, on the other hand, makes sense as a measure of the change of the tangent spaces of the hypersurface. Requiring that f preserves just the second fundamental forms is not enough for the extension. Fubini introduced a cubic form, now called Fubini's cubic form, and claimed that f can be extended if it preserves the second fundamental form and the Fubini's cubic form. One way to understand the second fundamental form and Fubini's cubic form is the following. A key object in projective geometry which can be regarded as playing the role of the metric in Euclidean geometry, is the set of lines in the projective space. A projective transformation sends a line to another line and this property characterizes projective transformations just as Euclidean motions can be characterized as isometries of the Euclidean space. Given a piece of hypersurface $S \subset \mathbf{P}^{n+1}$, choose a point $x \in S$. The tangent space of S at x is the information about which lines of $\mathbf{P}^{n+1}$ through x are tangent to S, or equivalently, osculate S to the 1st order. The second fundamental form (resp. Fubini's cubic form) of S at x is the information about which lines of $\mathbf{P}^{n+1}$ through x osculate S to the 2nd order (resp. 3rd order). Thus Fubini's claim is that if the 3rd order osculation data of lines are preserved under f then the map f can be extended to a projective transformation. Unfortunately Fubini's claimed proof was incomplete except in the case of $n = 2$. Years later, E. Cartan introduced the method of moving frames and gave a simple proof in the case of $n = 2$. It turns out that Cartan's method can be generalized to higher dimensions, and a complete proof of Fubini's claim using Cartan's method was obtained by Jensen and Musso in 1993 ([JM], also see the references therein for the works of Cartan and Fubini).

The notion of the second fundamental form and Fubini's cubic form depends how the piece of hypersurface is embedded in $\mathbf{P}^{n+1}$ and so the result is of extrinsic nature. However if we consider a complete hypersurface in $\mathbf{P}^{n+1}$, the situation changes drastically. For example, consider the following special case of Problem.

Let S, S' be two (complete) complex hypersurfaces in the complex projective space $\mathbf{P}^{n+1}$. Let $f : S \to S'$ be a biholomorphic map. When can we extend f to a projective transformation of $\mathbf{P}^{n+1}$?

The answer in this complete case is fairly simple. Any f can be extended to a projective transformation. As a matter of fact, if a compact complex manifold of dimension n can be embedded in $\mathbf{P}^{n+1}$, then the embedding is intrinsic in the sense that any two such embeddings are related by a projective transformation. (In fact, such a compact complex manifold when $n > 2$ has a unique line bundle whose first Chern class is a generator of $H_2(S, \mathbf{Z})$ and which has non-zero holomorphic sections. The holomorphic sections of the line bundle defines the embedding into $\mathbf{P}^{n+1}$.) Thus for the complete case, the essence of the result of Fubini-Cartan-Jensen-Musso is the following intrinsic extension theorem.

Let S, S' be compact complex manifolds which can be embedded in the complex projective space. The second fundamental form and Fubini's cubic form can be regarded as intrinsically defined tensors on S and S'. Let $U \subset S, U' \subset S'$ be connected open subsets and $\varphi : U \to U'$ be a biholomorphic map preserving the second fundamental form and the cubic form. Then φ can be extended to a biholomorphic map $\Phi : S \to S'$.

Recall that a complete hypersurface of $\mathbf{P}^{n+1}$ is the zero set of a homogeneous polynomial in the homogeneous coordinate of $\mathbf{P}^{n+1}$. The degree of this polynomial is called the degree of the hypersurface. When S and S' are hypersurfaces of degree $< n + 1$, there exist lines of $\mathbf{P}^{n+1}$ through x which lie on S for any given point $x \in S \subset \mathbf{P}^{n+1}$. In this case, one can show that the second fundamental form and the cubic form at x can be recovered from the information about the set of tangent directions of lines through x lying on S. But the notion of lines through x lying on S has an intrinsic meaning as follows. An n-dimensional compact complex manifold which can be embedded in $\mathbf{P}^{n+1}$ for $n > 2$, must be simply connected and $H_2(S, \mathbf{Z}) \cong \mathbf{Z}$, by Lefshetz hyperplane section theorem (e.g. [GH] p.156). Let X be a compact complex manifold which is simply connected and $H_2(X, \mathbf{Z}) \cong \mathbf{Z}$. A complex curve in S passing through a general point $x \in S$ is called a minimal rational curve if it is biholomorphic to $\mathbf{P}^1$ (with a mild singularity allowed) and its homology class in $H_2(S, \mathbf{Z})$ is minimal among those curves passing through x. When X is embeddable in $\mathbf{P}^{n+1}$ as a hypersurface of degree $< n + 1$, a minimal rational curve through x is precisely a line through x lying on X. Thus the result of Cartan-Fubini-Jensen-Musso implies the following.

Let S, S' be hypersurfaces of degree $< n + 1$ in $\mathbf{P}^{n+1}$. Let $U \subset S, U' \subset S'$ be connected open subsets and $\varphi : U \to U'$ be a biholomorphic map whose differential sends the tangent vector of a minimal rational curve of S to the tangent vector of a minimal rational curve on S'. Then φ can be extended to a biholomoprhic map $\Phi : S \to S'$.

Stated in this way, it is really a result on the intrinsic geometry of the manifold S and S'. I want to discuss a generalization of this. To generalize the above situation, we consider a smooth projective algebraic variety X (this means that it is a compact complex manifold which can be embedded in some projective space $\mathbf{P}^N$ for possibly very big N), which is simply connected and $H_2(X, \mathbf{Z}) \cong \mathbf{Z}$. Furthermore, we make the assumption that through each point $x \in X$, there exists a complex curve which is the image of a holomorphic map from $\mathbf{P}^1$. The last assumption may look too special, but, in fact, it is satisfied by a fairly large class of algebraic varieties, called Fano manifolds. In algebraic geometric terms, a Fano manifold is a smooth algebraic variety whose anti-canonical bundle, i.e., the determinant of the tangent bundle, is positive. In differential geometric terms, a Fano manifold is a compact Kähler manifold with positive Ricci curvature. A fundamental result of Mori ([Mo]) says that there exists a holomorphic map from $\mathbf{P}_1$ to X whose image contains any given point of X. The assumptions we made about X is simply that X is a Fano manifold with second Betti number 1. As before, we can consider minimal rational curves through $x \in X$. For a general $x \in X$, we define $\mathcal{C}_x \subset \mathbf{P}T_x(X)$ as the set of tangent vectors to minimal rational curves through x. Then a natural generalization of the above theorem is

Question *Let X, X' be Fano manifolds of second Betti number 1. Let $U \subset X, U' \subset X'$ be connected open subsets and $\varphi : U \to U'$ be a biholomorphic map sending $\mathcal{C}_x$ to $\mathcal{C}_{\varphi(x)}$ for general $x \in U$. Can φ be extended to a biholomorphic map $\Phi : X \to X'$?*

Since the problem originated from the works of Cartan and Fubini, we will call an extension of holomorphic map of the above kind as 'Cartan-Fubini type extension'. Recently, Ngaiming Mok and I have obtained an affirmative answer to the question under an additional assumption ([HM2]).

Theorem 1 *Cartan-Fubini type extension holds if* $\dim(\mathcal{C}_x) > 0$ *and* $\mathcal{C}_x$ *has finite Gauss map as a projective subvariety of* $\mathbf{P}T_x(X)$ *.*

The additional assumption is satisfied for many examples. It is satisfied for hypersurfaces of degree $2 \leq d \leq n-1$. For these cases, our proof is completely independent from the proof of Jensen-Musso. It is also satisfied for Hermitian symmetric spaces and more generally homogeneous Fano manifolds with second Betti number 1. In these two cases, related results were obtained by Ochiai ([Oc]) and Yamaguchi ([Ya]) by the method of Lie algebra cohomology. Again, our proof is independent of theirs. It is also satisfied by many complete intersections (transversal intersections of hypersurfaces). Also the moduli space of stable bundles over an algebraic curve of sufficiently high rank satisfies the condition.

Our proof uses the deformation theory of rational curves. It consists of two independent parts. The fist part is to show that φ sends local pieces of minimal rational curves in U to local pieces of minimal rational curves in U'. Namely, the information about the tangent directions can be integrated to give information about the curves. This first part uses the assumption about the Gauss map of $\mathcal{C}_x$. The second part is an extension theorem for maps which sends pieces of minimal rational curves to pieces of minimal rational curves. The second part does not use the assumption on the Gauss map and only uses the positive dimensionality assumption. Let me briefly explain the essential points of the proof.

For the first step, we need a definition from the theory of distributions. Recall (e.g. [Sp] I chapter 6) that a distribution D on a manifold Y is integrable, i.e., given by the tangent spaces to some foliation, if and only if the Frobenius bracket tensor $[,] : \wedge^2 D \to T(Y)/D$ is identically zero. The usual proof for C^∞ setting works for the holomorphic setting as well. Thus given a holomorphic distribution, i.e. a subbundle of the holomorphic tangent bundle, it is integrable, i.e., given by the holomorphic tangent spaces of some holomorphic foliation, if the holomorphic

Frobenius bracket tensor vanishes. For a given distribution, define the subdistribution $Ch(D)$, called the Cauchy characteristic of D, by

$$Ch(D)_x \quad := \quad \{v \in D_x, [v, w] = 0 \text{ for all } w \in D_x\}.$$

This is a distribution on an open subset of Y and it is always integrable.

Now let $\mathcal{C} \subset \mathbf{P}T(X)$ be the subvariety of the projectivized tangent bundle of the Fano manifold X whose fiber at $x \in X$ is exactly $\mathcal{C}_x$, the set of tangent directions to minimal rational curves through x. On the smooth part of $\mathcal{C}$, we define a holomorphic distribution $\mathcal{P}$ as follows. Given a smooth point $\alpha \in \mathcal{C}_x$, let $V_\alpha \subset T_x(X)$ be the subspace corresponding to the tangent space of $\mathcal{C}_x$ at α. Define

$$\mathcal{P}_\alpha \quad := \quad (d\pi)^{-1}_\alpha (V_\alpha)$$

where $d\pi : T_\alpha(\mathcal{C}) \to T_x(X)$ is the differential of the projection $\pi : \mathcal{C} \to X$ induced by the natural projection $\mathbf{P}T(X) \to X$. On $\mathcal{C}$, there exists a foliation $\mathcal{F}$ by curves, whose leaves are just lifts of the minimal rational curves to their tangent vectors. Then $\mathcal{F} \subset Ch(\mathcal{P})$ from basic deformation theory of curves. The key point of the proof of the first part is to show that $\mathcal{F} = Ch(\mathcal{P})$ by proving that any extra vector in the Cauchy characteristic must be killed by the kernel of the differential of the Gauss map. Once this is proved, then the first part is completed because $\mathcal{P}$ is determined by $\mathcal{C}$ so preserved by φ, thus $\mathcal{F}$ is preserved, which amounts to saying that local pieces of minimal rational curves are preserved.

After the first part is done, the second part of extending maps which send pieces of minimal rational curves to pieces of minimal rational curves is done by analytic continuation along minimal rational curves. For example, suppose C is a minimal rational curve which intersects U. We want to extend φ to points on C. The key tool here is so-called the bend-and-break technique of Mori ([Ko] Ch. II), that a family of minimal rational curves cannot contain two distinct points in common. Given a point $z \in C$ away from U, we choose a family $\{C_t, t \in \mathbf{C}, |t| < 1\}$ of minimal rational curves sharing the point z, which exists from the assumption on the positive dimensionality of $\mathcal{C}_z$. By φ, these curves are sent to a family of minimal rational curves $\{C'_t\}$ in X'. They have some common points because this is the case when $z \in U$ (a kind of algebro-geometric continuity argument). Now by the bend-and-break there exists a unique common point $z' \in X'$. We define the image of z to be z' and this gives the desired extension.

This way, we can extend the domain of the definition of φ to a larger open set U_1 by adding minimal rational curves intersecting U. Then we can continue by adding minimal rational curves intersecting U_1 and so on. A priori, this analytic continuation may define a multi-valued holomorphic map because a given point can lie on several different minimal rational curves. However the univalency can be proved by using the bend-and-break again.

There are many applications of Cartan-Fubini type extension of holomorphic maps. Let me just mention the following rigidity for holomorphic maps.

Theorem 2 *Let Y be any complex projective algebraic manifold of dimension n and X be a Fano manifold of dimension n of second Betti number 1 for which the Cartan-Fubini type extension holds. Then given any family of surjective holomorphic maps $f_t : Y \to X$, there exists a family of automorphisms g_t of X so that $f_t = g_t \circ f_0$.*

Theorem 2 is new even for hypersurfaces of degree $2 \leq d \leq n-1$. Its proof depends on our previous result [HM1] and I will omit it.

Finally, I would like to make some remarks about the additional condition of Theorem 1. First the Gauss map condition. This condition is not satisfied for the complex projective space and in fact, our Theorem 1 is false in that case. But it seems that excepting the complex projective space, all the other Fano manifolds with second Betti number 1 satisfy this condition. The condition on the positive dimensionality, on the other hand, fails for many examples, say for hypersurfaces of degree n or $n+1$. However it is conceivable that the Cartan-Fubini type extension still holds for these cases, too. As a matter of fact, for certain Fano 3-folds, we have checked that although $\dim(\mathcal{C}_x) = 0$, the Cartan-Fubini extension holds.

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