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CRYSTAL GRAPHS FOR BASIC REPRESENTATIONS OF QUANTUM AFFINE ALGEBRAS

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Abstract. We give a realization of crystal graphs for basic representations of quantum affine algebras in terms of new combinatorial objects called the Young walls.

1. Introduction

In [6, 7], Kashiwara developed the theory of *crystal bases* for integrable modules over the quantum groups associated with symmetrizable Kac-Moody algebras. The crystal bases can be viewed as bases at $q = 0$ and they are given a structure of colored oriented graph, called the *crystal graphs*, with arrows defined by Kashiwara operators. The crystal graphs have many nice combinatorial features reflecting the internal structure of integrable representations of quantum groups. For instance, the characters of integrable representations can be computed by counting the elements in the crystal graphs with a given weight. Moreover, the tensor product decomposition of integrable modules into a direct sum of irreducible submodules is equivalent to decomposing the tensor product of crystal graphs into a disjoint union of connected components. Therefore, to understand the combinatorial nature of integrable representations, it is essential to find realizations of crystal graphs in terms of nice combinatorial objects.

In [9], Misra and Miwa constructed the crystal graphs for basic representations of quantum affine algebras $U_q(A_n^{(1)})$ using Young diagrams with colored boxes. Their idea was extended to construct crystal graphs for irreducible highest weight $U_q(A_n^{(1)})$-modules of arbitrary higher level [2]. The crystal graphs constructed in [9] can be parametrized by certain paths which arise naturally in the theory of solvable lattice models. Motivated by this observation, Kang, Kashiwara, Misra, Miwa, Nakashima and Nakayashiki developed the theory of *perfect crystals* for general quantum affine algebras and gave a realization of crystal graphs for irreducible highest weight modules of arbitrary higher level in terms of *paths*. In this way, the theory of vertex models can be explained in the language of representation theory of quantum affine algebras and the 1-point function of the vertex model was expressed as the quotient of the string function and the character of the corresponding irreducible highest weight representation.

In [3], Kang gives a realization of crystal graphs for basic representations of quantum affine algebras of classical type using some new combinatorial objects

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which are called the *Young walls*. The Young walls consist of colored blocks that are built on the given *ground-state* and can be viewed as generalizations of Young diagrams. The rules for building Young walls are quite similar to playing with LEGO blocks and the Tetris game. The crystal graphs for basic representations are characterized as the set of all *reduced proper Young walls*. The weight of a Young wall can be computed easily by counting the number of colored blocks that have been added to the ground-state. Hence the weight multiplicity is just the number of all reduced proper Young walls of given weight.

In this article, we illustrate this construction of crystal graphs for the quantum affine algebra $U_q(C_2^{(1)})$. The readers can find more details in [1] and [3].

2. The quantum affine algebra $U_q(C_2^{(1)})$ and crystal bases

Let $I = \{0, 1, 2\}$ be the index set. Consider the generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$ of affine type $C_2^{(1)}$ and its Dynkin diagram:

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix} \quad \text{and} \quad \underset{0}{} \quad \underset{1}{} \quad \underset{2}{} .$$

Let $P^\vee = \mathbf{Z}h_0 \oplus \mathbf{Z}h_1 \oplus \mathbf{Z}h_2 \oplus \mathbf{Z}d$ be a free abelian group, called the *dual weight lattice* and set $\mathfrak{h} = \mathbf{C} \otimes_{\mathbf{Z}} P^\vee$. We define the linear functionals α_i and Λ_i $(i \in I)$ on $\mathfrak{h}$ by

$$\begin{aligned} &\alpha_i(h_j) = a_{ji}, \quad \alpha_i(d) = \delta_{0,i}, \\ &\Lambda_i(h_j) = \delta_{ij}, \quad \Lambda_i(d) = 0 \qquad (i, j \in I). \end{aligned}$$

The α_i (resp. h_i) are called the *simple roots* (resp. *simple coroots*) and the Λ_i are called the *fundamental weights*. We denote by $\Pi = \{\alpha_i | i \in I\}$ (resp. $\Pi^\vee = \{h_i | i \in I\}$) the set of simple roots (resp. simple coroots).

Let $c = h_0 + h_1 + h_2$ and $\delta = \alpha_0 + 2\alpha_1 + \alpha_2$. Then we have $\alpha_i(c) = 0$, $\delta(h_i) = 0$ for all $i \in I$ and $\delta(d) = 1$. We call c (resp. δ) the *canonical central element* (resp. *null root*). The free abelian group $P = \mathbf{Z}\Lambda_0 \oplus \mathbf{Z}\Lambda_1 \oplus \mathbf{Z}\Lambda_2 \oplus \mathbf{Z}\delta$ is called the *weight lattice* and the elements of P are called the *affine weights*.

We denote by q^h $(h \in P^\vee)$ the basis elements of the group algebra $\mathbf{C}(q)[P^\vee]$ with the multiplication $q^h q^{h'} = q^{h+h'}$ $(h, h' \in P^\vee)$. Set $q_0 = q_2 = q^2$, $q_1 = q$ and $K_0 = q^{2h_0}$, $K_1 = q^{h_1}$, $K_2 = q^{2h_2}$. We will also use the following notations.

$$[k]_i = \frac{q_i^k - q_i^{-k}}{q_i - q_i^{-1}}, \quad [n]_i! = \prod_{k=1}^{n} [k]_i, \quad \text{and} \quad e_i^{(n)} = e_i^n / [n]_i!, f_i^{(n)} = f_i^n / [n]_i!.$$

Definition 2.1. The *quantum affine algebra* $U_q(C_2^{(1)})$ of type $C_2^{(1)}$ is the associative algebra with 1 over $\mathbf{C}(q)$ generated by the symbols e_i, f_i $(i \in I)$ and q^h $(h \in P^\vee)$ subject to the following defining relations:

$$q^0 = 1, \quad q^h q^{h'} = q^{h+h'} \quad (h, h' \in P^\vee),$$
$$q^h e_i q^{-h} = q^{\alpha_i(h)} e_i, \quad q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i \quad (h \in P^\vee, i \in I),$$
$$e_i f_j - f_j e_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \quad (i, j \in I),$$
$$e_0^2 e_1 - (q^2 + q^{-2}) e_0 e_1 e_0 + e_1 e_0^2 = 0,$$
$$f_0^2 f_1 - (q^2 + q^{-2}) f_0 f_1 f_0 + f_1 f_0^2 = 0,$$
$$e_1^3 e_0 - (q^2 + 1 + q^{-2}) e_1^2 e_0 e_1 + (q^2 + 1 + q^{-2}) e_1 e_0 e_1^2 - e_0 e_1^3 = 0,$$
$$f_1^3 f_0 - (q^2 + 1 + q^{-2}) f_1^2 f_0 f_1 + (q^2 + 1 + q^{-2}) f_1 f_0 f_1^2 - f_0 f_1^3 = 0,$$
$$e_1^3 e_2 - (q^2 + 1 + q^{-2}) e_1^2 e_2 e_1 + (q^2 + 1 + q^{-2}) e_1 e_2 e_1^2 - e_2 e_1^3 = 0,$$
$$f_1^3 f_2 - (q^2 + 1 + q^{-2}) f_1^2 f_2 f_1 + (q^2 + 1 + q^{-2}) f_1 f_2 f_1^2 - f_2 f_1^3 = 0,$$
$$e_2^2 e_1 - (q^2 + q^{-2}) e_2 e_1 e_2 + e_1 e_2^2 = 0,$$
$$f_2^2 f_1 - (q^2 + q^{-2}) f_2 f_1 f_2 + f_1 f_2^2 = 0,$$
$$e_0 e_2 = e_2 e_0, \quad f_0 f_2 = f_2 f_0.$$

We call $(A, \Pi, \Pi^\vee, P, P^\vee)$ the *Cartan datum* associated with the quantum affine algebra $U_q(C_2^{(1)})$.

We now briefly review the basics of crystal basis theory. A $U_q(C_2^{(1)})$-module M is called *integrable* if

(i) $M = \bigoplus_{\lambda \in P} M_\lambda$ (resp. $M = \bigoplus_{\lambda \in \bar{P}} M_\lambda$), where

$$M_\lambda = \{v \in M \mid q^h v = q^{\lambda(h)} v \text{ for all } h \in P^\vee \text{ (resp. } h \in \bar{P}^\vee)\},$$

(ii) for each $i \in I$, M is a direct sum of finite dimensional irreducible U_i-modules, where U_i denotes the subalgebra generated by e_i, f_i, $K_i^{\pm 1}$ which is isomorphic to $U_q(\mathfrak{sl}_2)$.

Fix $i \in I$. By the representation theory of $U_q(\mathfrak{sl}_2)$, any element $v \in M_\lambda$ may be written uniquely as

$$v = \sum_{k \geq 0} f_i^{(k)} v_k,$$

where $v_k \in \ker e_i \cap M_{\lambda + k\alpha_i}$. We define the endomorphisms $\tilde{e}_i$ and $\tilde{f}_i$ on M, called the *Kashiwara operators*, by

$$\tilde{e}_i v = \sum_{k \geq 1} f_i^{(k-1)} v_k, \qquad \tilde{f}_i v = \sum_{k \geq 0} f_i^{(k+1)} v_k.$$

Let $\mathbf{A}$ be the subring of $\mathbf{C}(q)$ consisting of the rational functions in q that are regular at $q = 0$.

Definition 2.2.
(a) A free $\mathbf{A}$-submodule L of an integrable U_q-module M, stable under $\tilde{e}_i$ and $\tilde{f}_i$, is called a *crystal lattice* if $M \cong \mathbf{C}(q) \otimes_{\mathbf{A}} L$ and $L = \oplus_{\lambda \in P} L_\lambda$, where $L_\lambda = L \cap M_\lambda$.
(b) A *crystal basis* of an integrable module M is a pair (L, B) such that

(i) L is a crystal lattice of M,

(ii) B is a $\mathbf{C}$-basis of L/qL,

(iii) $B = \cup_{\lambda \in P} B_\lambda$, where $B_\lambda = B \cap (L_\lambda / qL_\lambda)$,

(iv) $\tilde{e}_i B \subset B \cup \{0\}$, $\tilde{f}_i B \subset B \cup \{0\}$,
(v) for $b, b' \in B$, $b' = \tilde{f}_i b$ if and only if $b = \tilde{e}_i b'$.

The set B is given a colored oriented graph structure by defining $b \quad {}^{i} \quad b'$ if and only if $b' = \tilde{f}_i b$. The graph B is called the *crystal graph* of M and it reflects the combinatorial structure of M. For instance, we have $\dim_{\mathbf{C}(q)} M_\lambda = \# B_\lambda$ for all $\lambda \in P$ (or $\lambda \in \bar{P}$). By extracting properties of the crystal graphs, we define the notion of abstract *crystals* as follows.

Definition 2.3 ([8]). An *affine crystal* is a set B together with the maps $\mathrm{wt} : B \to P$, $\varepsilon_i : B \to \mathbf{Z} \cup \{-\infty\}$, $\varphi_i : B \to \mathbf{Z} \cup \{-\infty\}$, $\tilde{e}_i : B \to B \cup \{0\}$, and $\tilde{f}_i : B \to B \cup \{0\}$, satisfying the following conditions:
(i) $\langle \mathrm{wt}(b), h_i \rangle = \varphi_i(b) - \varepsilon_i(b)$ for all $b \in B$,
(ii) for $b \in B$ with $\tilde{e}_i b \in B$, $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$,
$\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$,
(iii) for $b \in B$ with $\tilde{f}_i b \in B$, $\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$,
$\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$,
(iv) $b' = \tilde{f}_i b$ if and only if $b = \tilde{e}_i b'$ for $b, b' \in B$,
(v) $\tilde{e}_i b = \tilde{f}_i b = 0$ if $\varepsilon_i(b) = -\infty$.

The crystal graph of an integrable $U_q(C_2^{(1)})$-module is an affine crystal.

Let B_1 and B_2 be affine crystals. A *morphism* $\psi : B_1 \to B_2$ of crystals is a map $\psi : B_1 \cup \{0\} \to B_2 \cup \{0\}$ such that
(i) $\psi(0) = 0$,
(ii) if $b \in B_1$ and $\psi(b) \in B_2$, then $\mathrm{wt}(\psi(b)) = \mathrm{wt}(b)$, $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, $\varphi_i(\psi(b)) = \varphi_i(b)$,
(iii) if $b, b' \in B_1$, $\psi(b), \psi(b') \in B_2$ and $\tilde{f}_i b = b'$, then $\tilde{f}_i \psi(b) = \psi(b')$.

The *tensor product* $B_1 \otimes B_2$ of B_1 and B_2 is the set $B_1 \times B_2$ whose crystal structure is defined by

$$\begin{aligned}
\mathrm{wt}(b_1 \otimes b_2) &= \mathrm{wt}(b_1) + \mathrm{wt}(b_2), \\
\varepsilon_i(b_1 \otimes b_2) &= \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle \mathrm{wt}(b_1), h_i \rangle), \\
\varphi_i(b_1 \otimes b_2) &= \max(\varphi_i(b_2), \varepsilon_i(b_1) + \langle \mathrm{wt}(b_2), h_i \rangle), \\
\tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \\
\tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}
\end{aligned}$$

3. The Young walls

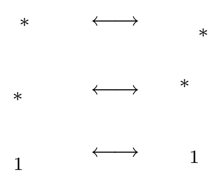
In this section, we will explain the notion of Young walls. The Young walls will be built of three kinds of *blocks*; the 0-block (0), the 1-block (1), and the 2-block (2). They are supposed to be colored by elements from the index set I and we do not allow rotations of the blocks. The 0-block and 2-block are of unit height, unit width, and half-unit thickness. The 1-block is of half-unit height, unit width, and unit thickness. With these blocks, we will build a wall of unit thickness, extending infinitely to the left, like playing with LEGO blocks. The base of the wall may not be arbitrary, but must be chosen from one of the following three.

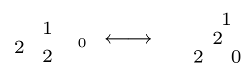
Y_{Λ_0} =

Y_{Λ_1} =

Y_{Λ_2} =

The drawings are meant to extend infinitely to the left. The dotted lines in the front parts of Y_{Λ_0} and Y_{Λ_2} signify where other blocks that will build up the wall of unit thickness may be placed. These will be called the *ground-state walls*. As it is quite awkward drawing these, and since we can't see the blocks lying to the back, we will simplify the drawing as follows. The drawing on the right shows an example.



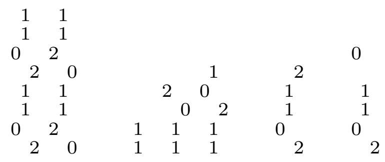


We will now list the rules for building the wall with colored blocks.

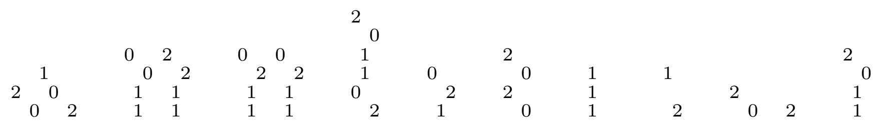
Rules 3.1.
(i) The wall must be built on top of one of the ground-state walls.
(ii) The blocks should be stacked in columns. No block may be placed on top of a column of half-unit thickness.
(iii) The top and the bottom of the 0-block and 2-block may only touch a 1-block. The sides of the 0-block may only touch a 2-block and vice versa.
(iv) Placement of 0-block and 2-block to either the front or the back of the wall must be consistent for each column. The 0-block and 2-block should always be placed at heights which are multiples of the unit length.
(v) Stacking more than two 1-blocks consecutively on top of another is not allowed.
(vi) Except for the right-most column, there should be no free space to the right of any block.
(vii) In the right-most column of a wall built on Y_{Λ_1}, the 0-block should be placed to the front and the 2-block should be placed to the back.

We will give some examples illustrating the rules for building the walls. From now on, the ground-state wall extending infinitely to the left will be omitted and what remains will be shaded in the drawings.

Example 3.2. Good walls.



Example 3.3. Bad walls.



A column in the wall is a *full column* if the height of the column is a multiple of the unit length and the top of the column is of unit thickness. The first wall in Example 3.2 has two full columns. The right-most column in the second wall is not full but the other two are. Neither the third nor the forth wall contain full columns.

Definition 3.4.

(a) A wall satisfying Rules 3.1 is called a *Young wall* of *ground state* Λ_i, if it is built on the ground state wall Y_{Λ_i} and the height of the columns are weakly decreasing as we go to the left.

(b) A Young wall is said to be *proper* if no two full columns of the wall are of the same height.

4. The crystal structure

In this section, we will define a crystal structure on the set of all proper Young walls. The action of Kashiwara operators will be described in a way similar to playing the Tetris game.

Definition 4.1.

(a) A block in a proper Young wall is *removable* if the wall remains a proper Young wall after removing the block.

(b) A place in a proper Young wall, where one may add a block to obtain another proper Young wall is called an *admissible slot*.

(c) A column in a proper Young wall is said to contain a *removable* δ, if we may remove a 0-block, a 2-block, and two 1-blocks from the column in some order and still obtain a proper Young wall.

(d) A proper Young wall is *reduced* if none of its columns contain a removable δ.

The second and the last wall of Example 3.2 are reduced. The first wall is not proper and the third wall contains one removable δ. The set of all reduced proper Young walls of ground-state Λ_i is denoted by $\mathcal{Y}(\Lambda_i)$.

We now defined the action of the Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$. A column in a proper Young wall is called *i-removable* if the top block of that column is a removable i-block. A column is *i-admissible* if the top of that column is an i-admissible slot. The action of Kashiwara operators is defined as follows.

1. Go through each column and write a 0 under each i-admissible column and a 1 under each i-removable column.

2. If we are dealing with the $i = 1$ case, there could be columns from which two 1-blocks may be removed. Place 1 1 under them. Under the columns that are 1-removable and at the same time 1-admissible, place 1 0. Write 0 0 under the columns that are twice 1-admissible.
3. From the (half-)infinite list of 0's and 1's, cancel out each 0 1 pair to obtain a finite sequence of 1's followed by some 0's (reading from left to right).
4. For $\tilde{e}_i$, remove the i-block corresponding to the right-most 1 remaining. Set it to zero if no 1 remains.
5. For $\tilde{f}_i$, add an i-block to the column corresponding to the left-most 0 remaining. Set it to zero if no 0 remains.

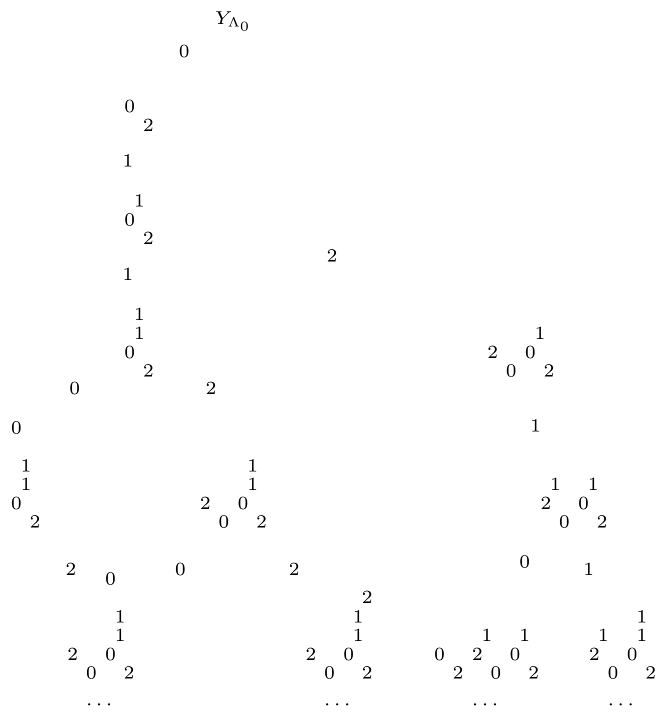
With this definition of Kashiwara operators, the crystal graphs for basic representations of $U_q(C_2^{(1)})$ are realized as the affine crystal consisting of all reduced proper Young walls.

Theorem 4.2. [1]
(*a*) *The set of all reduced proper Young walls is stable under the Kashiwara operators defined above. That is, if Y is a reduced proper Young wall, then $\tilde{e}_iY$ and $\tilde{f}_iY$ are either reduced proper Young walls or 0 for all $i \in I$.*
(*b*) *For each $i = 0, 1, 2$, we have the isomorphism of crystals*

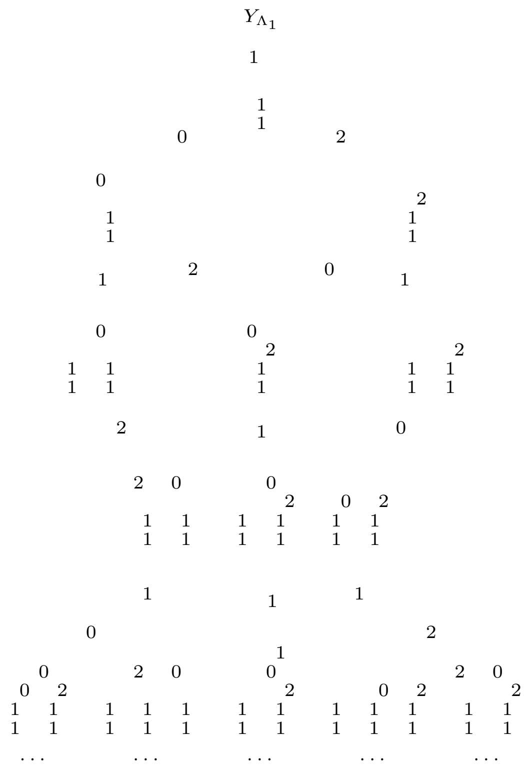
$$\mathcal{Y}(\Lambda_i) \cong B(\Lambda_i).$$

In the next examples, we draw the crystal graphs $B(\Lambda_0)$ and $B(\Lambda_1)$ in terms of reduced proper Young walls.

Example 4.3. Crystal graph of $B(\Lambda_0)$.



Example 4.4. Crystal graph of $B(\Lambda_1)$.



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